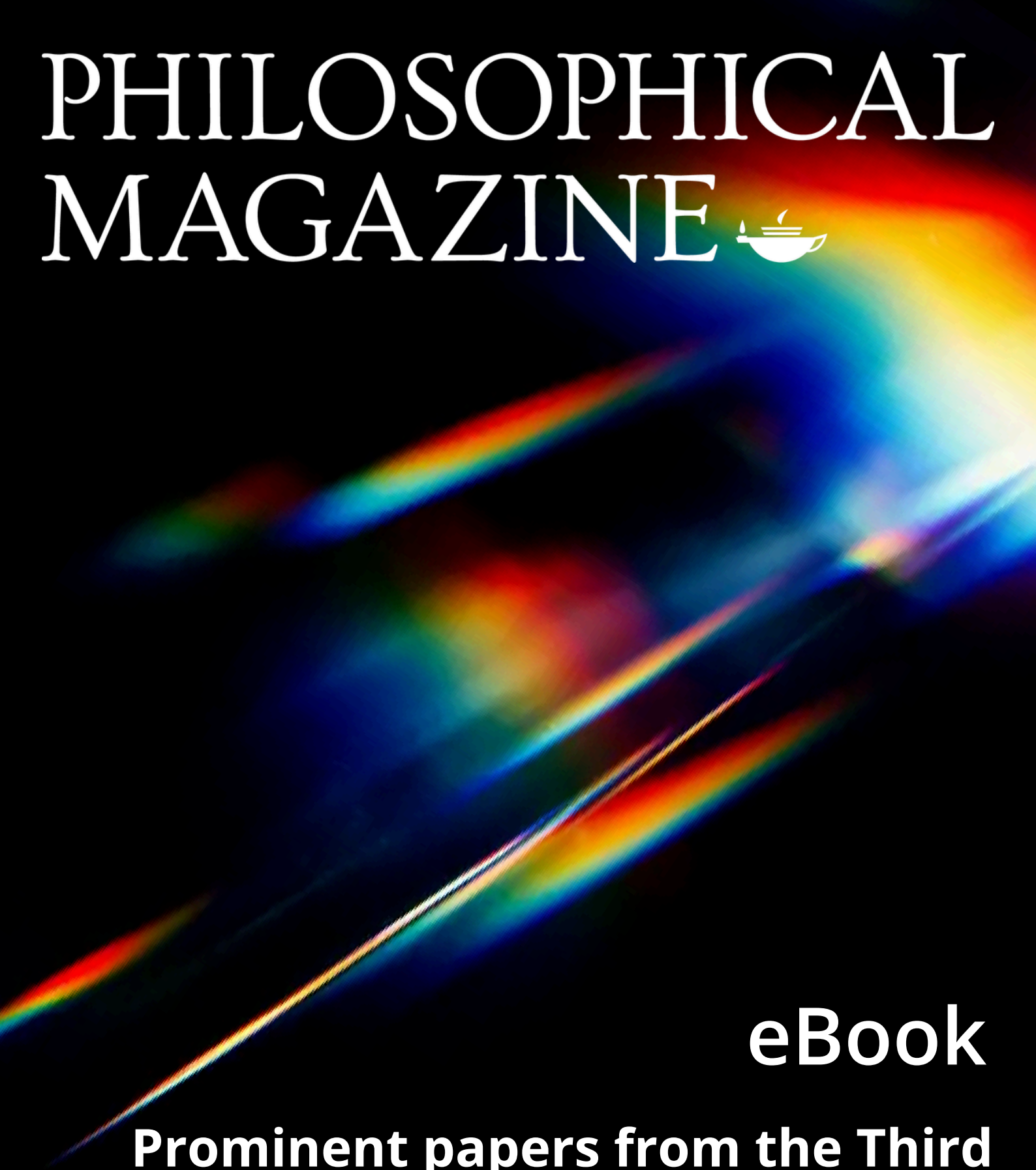
PHILOSOPHICAL MAGAZINE



eBook

Prominent papers from the Third Baron Rayleigh: Volume 1

The scattering of light by small particles

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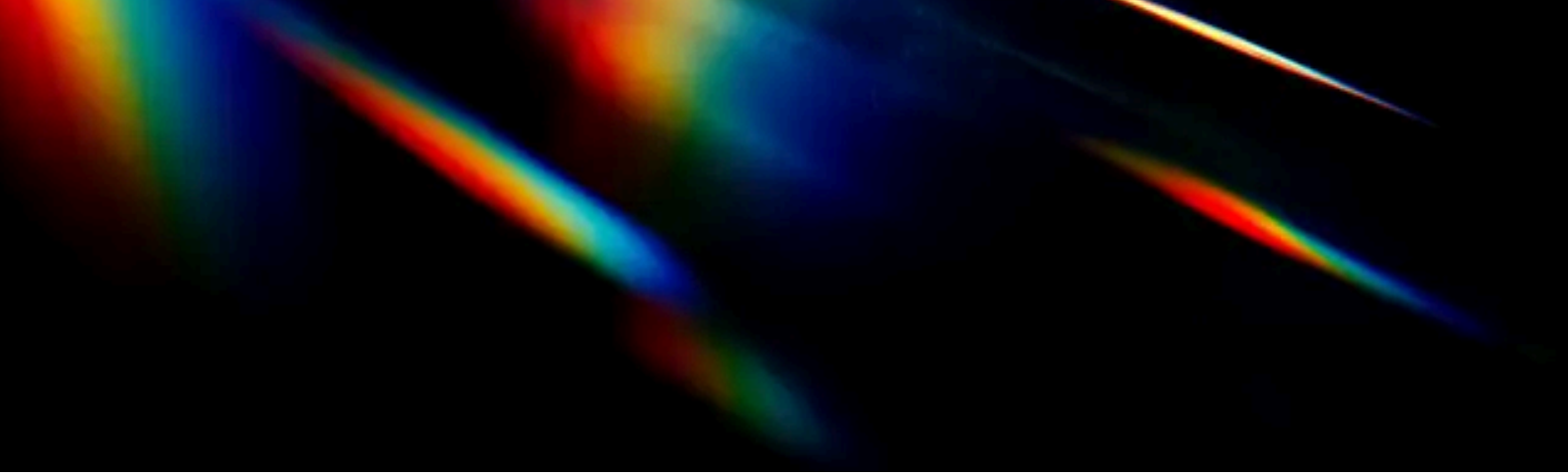
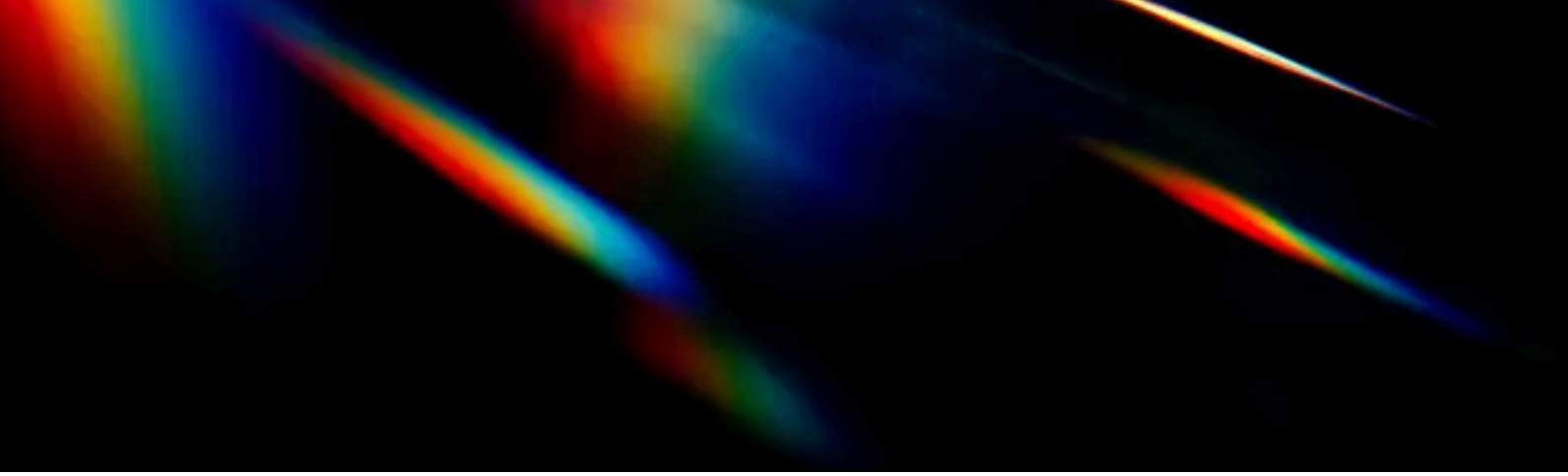


Introduction

The Third Baron Rayleigh (1842-1919) was one of Britain's great physicists, who made original contributions to all branches of classical physics known in his day. He was born John William Strutt and, of the 446 papers he wrote, the first 20 or so were published under that name, prior to his inheriting the title Lord Rayleigh on the death of his father in 1873.

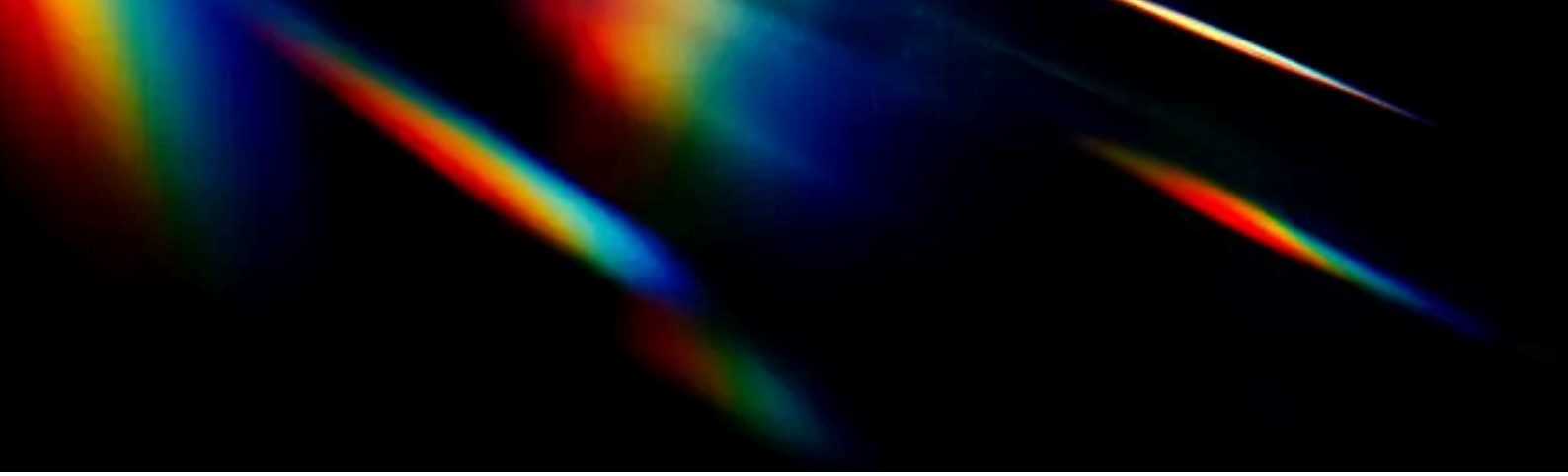
That John William Strutt was destined to greatness was evident from his graduation as Senior Wrangler (i.e. top man) in the Mathematics Tripos in 1865 from Trinity College, Cambridge, where he was subsequently elected to a Fellowship. He learnt German and his reading of Helmholtz's 'Die Lehre von den Tonempfindungen' (The Sensation of Tone) sparked his interest in acoustics and the physiology of hearing. It also led him to Helmholtz's work on colour vision, a subject also studied by James Clerk Maxwell – another great British physicist whom Strutt was destined to follow as Professor of Experimental Physics at the Cavendish Laboratory in 1879.

Strutt's marriage to Evelyn Balfour in 1871 meant he had to resign his Trinity Fellowship, which was for bachelors only, and so he returned to the family estate in Terling, Essex. There he arranged for the conversion of an old stable and hay loft adjoining the main house for use as his private laboratory. However, within a year he suffered a severe attack of rheumatic fever. To assist his recovery, he visited Egypt and, during a trip down the Nile, wrote the majority of the volume 1 of 'The Theory of Sound'.



This famous and heavily mathematical text, which was eventually published in 1877, is still a reference work for today's acoustic engineers.

Strutt wrote his first paper in 1869 'On some Electromagnetic Phenomena considered in connexion with the Dynamical Theory', which was published in the Philosophical Magazine – a journal that was to become his favourite repository for about 200 of the papers he was eventually to write. His early work was theoretical in nature and he developed mathematical techniques which he applied to solve analogous problems in dynamics, acoustics, optics and elasticity. He was, however, an experimentalist too and it was while repeating some of Maxwell's experiments using a colour top, that he realized the importance of background illumination and in particular the degree of blueness of the sky when making observations in daylight. At this time James Tyndall, working at the Royal Institution, had shown that the passage of electricity through certain organic vapours containing fine particles gave rise to a blue light attributable to scattering. Strutt saw the analogy here with the blue colour of the sky and developed a classic theory which predicted that light incident upon an array of particles with size smaller than the wavelength of the light would be scattered in inverse proportion to the fourth power of the wavelength. Thus, the blue light from the sun is scattered more than the red and it is this light we see emanating from the sky. In the evening the less-scattered red light is seen directly from a setting sun. His theory also predicted the optical polarization of the scattered sunlight.



Strutt's classic paper 'On the Light from the Sky, its Polarization and Colour' was published in Philosophical Magazine in 1871. This paper, and subsequent related ones, are included in this first of several volumes devoted to the works of Lord Rayleigh. However, Volume 1 starts with two recent articles written by the present compiler of this series of eBooks, in order to provide background to the topic and put the following historical papers in context.

Author: *Professor Edward A. Davis*

Professor Edward A. Davis is a materials scientist with a passion for studying the history of physics. During his career, Davis has published over 200 papers – many with his research students – and been an author, co-author or editor of seven books. Since retirement, Davis has been granted the position of Distinguished Research Fellow in the Department of Materials Science and Metallurgy at the University of Cambridge (UK).

The colour and polarisation of light from the sky: Part I

Edward A Davis
Department of Materials Science and Metallurgy, University of Cambridge

The Third Baron Rayleigh is rightly credited for explaining in detail why the sky is blue and the light from it polarized. A succession of papers in the *Philosophical Magazine* – the first published in 1871 and the last in 1918 – developed the theory of scattering of light by particles of size small in comparison to its wavelength, a theory that now bears Rayleigh's name. However, the narrative leading up to this explanation has its beginnings in the musings of Greek philosophers, the development of these by the insights of Arab Scholars and the observations of painters, and more latterly in the theories and experiments of scientists. An historical study of the topic provides a fascinating insight into how scientific discoveries and explanations evolve with the passage of time and with the continual acquisition of new knowledge.

Note: Part I covers the period up the work of John Tyndall. Part II, starting with John William Strutt (later the Third Baron Rayleigh), will be published in the next issue of the Newsletter.

Greek Philosophers and Arab Scholars

When considering early explanations for the sky's colour, it is important to note that from the time of Aristotle (350 BC) until Newton's famous prism experiment (1666 AD), colours were considered to arise from mixtures of black and white present in different proportions. Thus, according to Aristotle's *On the Senses*, yellow was deemed to consist of about one-sixth black and five-sixths white; red, one-third black and two-thirds white; purple, equal proportions of black and white; green, two-thirds black and one-third white, blue, five-sixths black and one-sixth white. Black and white themselves were considered by Aristotle to be invisible. Greek philosophers frequently referred to brightness rather than colour (luminosity rather than hue) when describing the appearance of objects: for example, blue is considered to be a dark colour, while yellow is a light one.

An early reference to the colour of the sky was made by one of Aristotle's students, Theophrastus, who is believed to be author of the work *On Colours*. In this account Theophrastus wrote:

'Air seen close at hand appears to have no colour, for it is so rare that it yields and gives passage to the denser rays of light, which thus shines through it; but when seen in a deep mass it looks practically dark blue. This again is a result of its rarity, for where light fails the air lets darkness through.'

Of importance here is the realisation by Theophrastus that the sky is not a fixed canopy lying at a great height above our heads – a concept that one might have thought early civilisations would have considered to be almost self-evident – but instead is associated here with the '*air seen close at hand*' and extending upwards from the Earth's surface. The blue arises because darkness overcomes the light, which is an extension of Aristotle's idea that blue is a mixture of black and white with the former (darkness) predominating in this case.

The works of classical Greek philosophers were for the most part lost during the Dark Ages but rediscovered by Arab scholars in the ninth century. One of these scholars, Abu Yusuf al-Kindi, who was born in present-day Iraq around 800, modestly exclaimed that he and his contemporaries could not hope to achieve anything like the teachings of the Greeks. Nevertheless, he went on to write what is the oldest known and preserved account addressed solely to the matter of *'the cause of the azure-blue colour which is considered to be the colour of the sky*'. Perhaps based on his familiarity with sandstorms, he proposed that *'earthly particles'*, are stirred upwards, creating a haze in the atmosphere. In al-Kindi's own words:

'The air surrounding the Earth gets weakly lighted by the earthly particles dissolved in it and changed into fiery ones due to the heat which they have accepted from the reflection of rays off the Earth. The shadowy air above us is visible because the light of the Earth and the light of the stars intermingle into a colour in the middle of shadow and light and this is the azure-blue colour.'

It was in 995 that the first scientifically significant observation of the sky was made. It involved nothing more than climbing a mountain. Abu Rayhan al-Biruni, an early astronomer in today's Afghanistan, climbed the highest mountain in Persia, Demavend at 5,610 metres. He reported that the clear sky was darker than it was when seen in the plains below. This important observation was consistent with the earlier idea that the colour of the sky had somehow to involve darkness as well as light. This idea was articulated by a 13th century teacher of Islamic law at a Quranic school in Cairo and amateur scientist, Ahmed al-Qarafi. In a treatise, he wrote:

'One does not see the sky at all, and what we don't see, we see as dark. Ask a blind man, what is it that you see, and he will answer "black darkness". And so the sky is also black, and under it is the air, which is transparent and luminous. Our gaze penetrates the air and sees it against the sky so to speak…… Thus the blue colour results from the purity of the air and the darkness of the sky, for we are dealing here with the mixture of the black and the pure.'

Note that al-Qarafi appears to invoke the Aristotelian theory of colours, namely as all colours being a *mixture* of black and white in different proportions, but there is a subtle difference in that in this account the blue arises from observing a black background *through* a white (pure) one. Also implicit here, although not referred to by al-Qarafi, is an explanation for the darkening of the sky with altitude as observed by al-Biruni. Although a-Qarafi did not propose any explanation for the air's luminosity during daylight, his basic concept of a black background seen through a mass of essentially transparent but illuminated air is a fairly accurate description of what is in fact seen when looking skywards.

Roger Bacon

The first Englishman to contribute to an understanding of the sky's colour was Roger Bacon (1220-1292), who became a Franciscan monk at the age of 27. Bacon's enthusiasm for the teachings of Aristotle was curtailed by the prevailing doctrine of the church, which forbade teaching of Aristotelian philosophies on account of them being regarded as contradictory to Christian beliefs in miracles, the creation, and other matters of doctrine. Building on the works of Aristotle and Arab scholars, which had suggested that the colour of the sky depended on a competition between black and white, and hence implicitly on the amount of air through which the dark background is viewed, Bacon's writings (which were secretly copied and circulated in Europe) included the proposal that the colour of air depended explicitly on its depth. He likened the air to water, with both media being transparent over short depths but getting progressively darker with distance. In his *Opus Maius* [1], Bacon writes

'I say here that the air, or the sphere of fire, or the heavens, near and remote, is of similar rarity as far as perception is concerned; but it has, however, some density of its own nature, and this density is able in a great distance to terminate the species of vision, which it cannot do in a short distance, and therefore it will be quite visible at a distance, but not near at hand.'

Bacon attributes the blue colour to traces of particles in the air. However, unlike al-Kindi, he does not invoke foreign bodies, but instead attributes the colouration to the nature of air itself. Nevertheless, he did not depart from Aristotle's theory of colours in which blue is considered close to black. He writes:

'Why a colour appears approaching black, namely blue, is explained in the same way as in the case of deep water, where in a like manner that colour appears owing to shadows projected by particles. Darkness is caused by these shadows, which is similar to blackness. This is what takes place in the air or medium between us and the last heaven.'

In the 13th century, the colour blue was not readily available as a pigment for artists to use on their canvases. Blue pigments were normally obtained by grinding minerals such as lapis lazuli or azurite and mixing them with oil. On account of their rarity and cost, their use was often limited, for example in religious paintings, to the robe of the Virgin Mary. A trick that some painters employed was to apply a translucent black pigment over a white base coat. Simply *mixing* black and white merely produces grey, whereas the technique of 'overlaying' can produce visual effects that are often blue in colour, contrary to expectations. A contemporary of Bacon was Ristoro d' Arezzo, who, drawing an analogy between the colour of the sky and the painter's pallet, wrote:

'Clever painters who paint in colour – when they want to get the colour blue – mix two different colours together: light and dark, and from this mixture, blue results. When I look at the sky, I see two opposite colours mixed, namely light and dark, and this is due to the air's depth.'

The argument used by d' Arezzo to refute the suggestion by Bacon that the air itself might be tinged blue – at first sight a convincing one – was that anything we see beyond the atmosphere, such as the sun, moon, planets and stars, should all take on a blue colouration, which, he exclaimed, is obviously not the case.

Leonardo da Vinci

Leonardo da Vinci was both a painter and a natural philosopher. In about 1510, he completed one of his codices – manuscripts containing, in tiny scribbled mirror writing, his thoughts on a wide variety of natural phenomena and mechanical inventions, including, for example, water pumps and flying machines. The one that contains his theories, and even the results of experiments, on the nature of the sky is known as the Leicester Codex, named after the first Lord of Leicester, Thomas Coke, who purchased it in 1717 from an Italian painter. It was re-sold in 1980 to an American, Armand Hammer, and is currently owned by Bill Gates (founder of the Microsoft Corporation) who purchased it at auction for over $30 million in 1994.

Leonardo's interest in the accurate representation of landscapes in paintings led him to prescribe (as an artist):

'….in painting, to make one (object) seem more distant than the other, it would be necessary to represent the air as a little thick, for you know that in air like this, the farthest things seen in it, such as mountains, seem blue – almost the colour of the air when the sun is in the east – because of the great quantity of air that is found between your eye and the mountains.'

In addition, he proposed (as a scientist):

'I say that the blue that is seen in the air is not its own colour, but is caused by the heated moisture having evaporated into the most minute imperceptible particles, which the beams of the solar rays attract and cause to seem luminous against the deep intense deep intense darkness of the region of fire that forms a covering above them.' [see reference 2]

'As a further example of the colour of the air, we may take the case of the smoke produced by old dry wood, for it comes out of the chimneys it seems to be pronounced blue when seen between the eye and a dark space, but as it rises higher and comes between the eye and the luminous air, it turns immediately to an ashen grey hue, and this come to pass because it no longer has darkness behind it.'

Leonardo also found that water blown in the form of spray into a dark space, through which solar rays pass, produces the same blue colouration. Another experiment (with strong connections to his main profession as a painter) was to stain a board with very strong black and overlay this with a thin transparent white. He who does this will, in his own words in the Codex, *'perceive that the lustre of the white will nowhere display a more beautiful blue than over the black, but it must be very thin and finely ground.'*

Isaac Newton

Winding forward by about 150 years to the time of Newton, we reach his ground-breaking experiment with a glass prism which he performed in 1666. Utilising sunlight entering through a hole in the shutter of a window at his home in Woolsthorpe, he positioned the prism to refract the light onto the opposite wall. The resulting oblong pattern contained all the colours of the visible spectrum from red to violet. Newton had, at a stroke, demolished Aristotle's theory that colours are mixtures of black and white by demonstrating that white light itself can be decomposed into all known colours. Although others before Newton had observed coloured fringes generated by glass and by thin layers of liquids, these had been dismissed as chromatic flaws or artefacts, without their significance being realised. Likewise, of course, rainbows had been observed since the dawn of man. Aristotle himself had a fairly detailed theory of rainbow formation, which (correctly) involved reflection from raindrops but which

(incorrectly) described its colours (of which he recognised only three: red, green and violet with yellow being dismissed as an 'illusion') in terms of combinations of light and dark.

In his book *Opticks*, the 2nd edition of which was published in 1718, Newton presented his colour circle in which the seven basic colours of the spectrum are arranged around its circumference. From this it is possible to predict the colour (hue) that results from mixing any two colours, in addition to its brightness (purity or saturation), with white being at the centre of the circle.

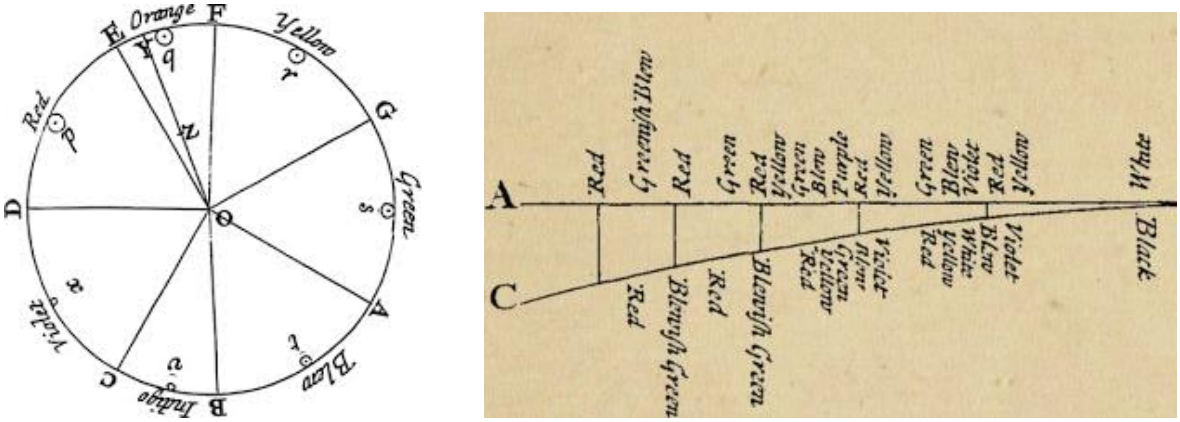


Figure 1. On the left, Newton's colour circle. The interior letters denote musical notes as Newton sought an analogy between these and the light spectrum. The right diagram shows various orders of colours observed in Newton's rings and recorded in detail in his book *Opticks*.

Even more consequential than his prism experiment for an understanding of the nature of light were Newton's observations on the coloured rings seen when two convex lenses were pressed together. Knowing the focal lengths of the two lenses, Newton calculated their radii of curvatures and hence their separation as a function of distance from their central point of contact. He thereby determined the difference in the gaps between the lenses corresponding to two adjacent coloured bands to be as small as 1/89,000 inch. The regularities of the rings and their appearance in both reflection and transmission suggested to Newton that light must have an intrinsic periodic property, the characteristic length of which was related to the refrangibility of the light rays, i.e. the amount by which they were refracted or, alternatively, their colour. It would seem that observations of these rings caused Newton to question his belief in the corpuscular theory of light. It also led him to suggest that the colour of objects could depend on the size of particles of which they were composed. The idea was that particles either reflect, refract or transmit light of a particular colour depending on their size.

In addition to using this concept to account for the colours of foam, paper, gold and copper, Newton applied it to the colour of the sky. The innermost circle of Newton's rings (as they are now called) is blue, which he regarded as the colour of least refrangibility. He called this 'the blue of the first order' and associated it with the smallest of particles. In his own words:

'The blue of the first order, though faint and little, may possibly the colour of some substances; and particularly the azure colour of the skies seems to be of this order. For all vapours when they begin to condense and coalesce into small parcels, become first of that bigness, whereby such an azure must be reflected before they can constitute clouds of other colours.'

So, Newton's explanation for the blue colour of the sky was, in essence, reflection from small water droplets in the atmosphere – a proposal that, as mentioned earlier, had been first suggested by Leonardo da Vinci.

Leonhard Euler and Horace Bénédict de Saussure

The Swiss mathematician and scientist, Leonhard Euler, wrote in 1760 that '*the cause of the visibility of objects is a motion of vibration extremely rapid, by which the minuter particles of their surfaces are agitated, and that the frequency of their vibrations determines the colour.*' Euler's study of

meteorological phenomena led him to assert that the higher one ascends from the earth's surface, the rarer the air becomes and the blue of the sky fades away into utter blackness. He then concludes that the air cannot be a perfectly transparent medium. He espoused the idea that '*the air is loaded with a great quantity of small particles which are not perfectly transparent, but which, being illuminated by the rays from the sun, receive from them a motion of vibration, which produces new rays proper to those particles; or else they are opaque and become visible to us from being illumined*'. A few years later he contended that the particles were not impurities but '*minute parcels of air'* itself. This description of the light from the sky anticipates, rather remarkably, the explanation to be offered by Rayleigh over 100 years later.

Another Swiss scientist, Horace Bénédict de Saussure, was both a geologist and Alpine explorer. He is credited with the invention of several instruments, including a hygrometer based on a single human hair, an anemometer, an electrometer (possibly the first) and what he called a cyanometer, a device for comparing the various degrees of blueness of the sky. The cyanometer was a circular card divided into 52 segments, each numbered and coloured with a different shade of blue varying from very pale to very dark [3]. It was simply held up to the sky and the closest colour match found by visual inspection (see Figure 2). As an experienced mountaineer, de Saussure was able to confirm Euler's darkening in the colour of the sky with altitude. He also chronicled variations in the degree of blueness with angle of observation from the zenith to the horizon and changes associated with atmospheric purity and meteorological conditions.

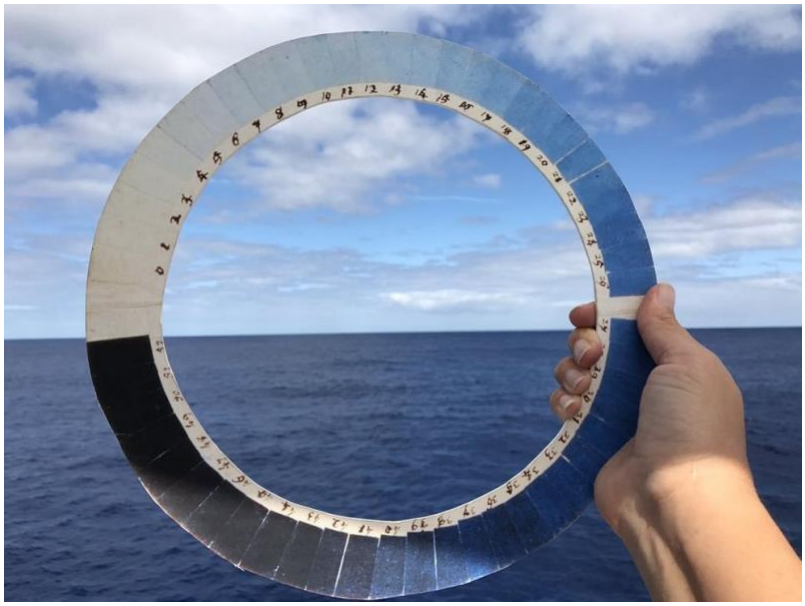


Figure 2. Replica of de Saussure's cyanometer

Alexander von Humboldt

Humboldt was a German naturalist and explorer. In 1802 he and a colleague climbed the mountain Chimborazo in Ecuador where a cyanometer was used in an attempt to quantify the darkening of the sky with altitude and determine the possible influence of oxygen on its depth of colour. The composition of the atmosphere had been determined about 30 years earlier by Joseph Priestley, Antoine Lavoisier and Henry Cavendish but it was not clear whether this changed with altitude. In 1804, Humboldt analysed samples of air obtained by Biot and Gay-Lussac during a balloon ascent to 23,000 feet. He obtained values of 21 percent oxygen, 78.7 percent nitrogen and 0.3 percent carbonic acid, the latter not having been found earlier. Furthermore, he found essentially no change in these values with altitude.

The wave nature of light and polarisation

Thomas Young was a physician and scholar who was appointed professor of natural philosophy at the Royal Institution in London in 1801. His proposal that light propagated in the form of periodic waves was demonstrated by displays of analogous phenomena with water waves in a so-called ripple tank. With this apparatus he was able to display interference phenomena, in particular those generated by

waves after passing through apertures. His famous optical double-slit experiment was conducted using light of various colours and, by noting that the spacing of the interference fringes changed with the colour of the light, he came to the conclusion that different colours were associated with different wavelengths. Young went on to determine these wavelengths and established that the visible spectrum ran from about 400 nm for violet light to 750 nm for red light. Furthermore, in contrast to Huygens who earlier had suggested light propagated as waves similar to longitudinal sound waves, Young was the first to propose that light waves were transverse in nature.

This significant proposal was vital for understanding the polarisation of light, a phenomenon discovered by a French physicist, Louis Malus, in 1808. The capability of crystals of Iceland spar to split a ray of light into two paths, named by Huygens an *ordinary* ray which obeyed the laws of refraction and an *extraordinary* one that did not, had been known for over 100 years before Malus's discovery. Using another double-refracting crystal, tourmaline, Malus was surprised to find, on viewing the reflection of sunlight from a pane of glass, that, at a particular orientation of the crystal and viewing angle, there was only one image, instead of the expected two. The result suggested that light itself was capable of having a particular orientation associated with it. Malus coined the word polarisation to identify this property but, as an adherent to the corpuscular theory of light, found it difficult to conceive how particles could exhibit it. Augustin Fresnel, a contemporary of Malus, recognised that Young's wave theory of light easily allowed for different orientations because the waves could, in principle, vibrate in any specified plane.

The experiments of Malus had not only provided insight into the nature of the two rays produced by double-refracting, or birefringent, crystals, but had also demonstrated that light reflected at a particular angle from glass was completely polarised. The angle of incidence at which this occurs is known as Brewster's angle, named after the Scottish physicist, David Brewster. It depends on the refractive index of the material; for glass it is 56 degrees and for water 53 degrees.

It was soon discovered that the light from the daytime sky is polarised too. Francis Arago noted how the polarisation pattern in the sky changed according to the sun's elevation and found that at 90° to the sun's direction it has a degree of polarisation that is as high as 75%. Arago also discovered that in a direction near to the horizon opposite to the setting sun the polarisation vanished – the so-called Arago point. Subsequently three other similar 'neutral' points have been identified (see Figure 3).

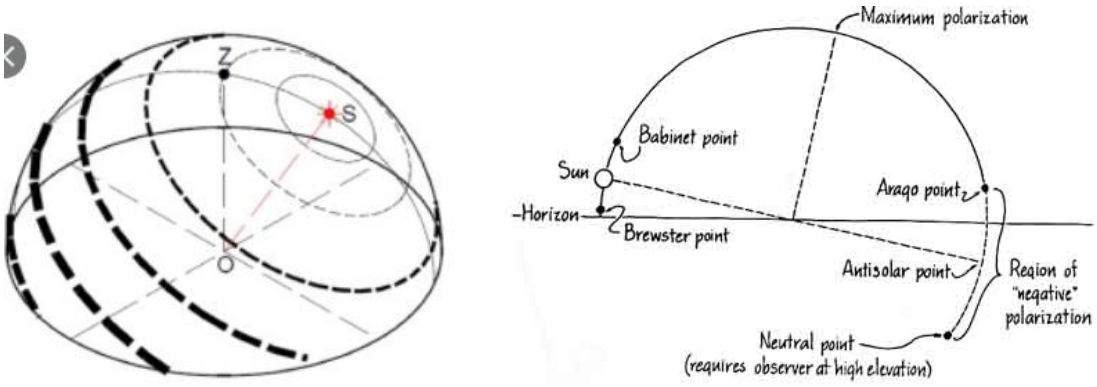


Figure 3. The light from the sky is polarised to varying degrees depending on the angle relative to the sun. It is most highly polarised along an arc at 90° to the sun's direction, with the electric vector along that arc (thickest line in diagram on the left). The degree of polarisation is zero at the neutral points as illustrated on the right diagram.

That the maximum polarisation of the daytime sky occurs along an arc subtending the observer and forming a plane at an angle of 90° from the sun's direction provided a challenge. The observation was not unrelated to the blue colour because it was soon discovered that, when the sky was cloudy and white, the extent of polarisation decreased substantially. It was clear that any explanation of the blue colour of the light from the sky must also account for its polarisation.

Newton's proposal that the blue colour of the sky arises from reflection from water droplets or ice crystals could be dismissed on the grounds that application of Brewster's law to reflection from water with a refractive index of 1.33 leads to the maximum polarisation occurring at an angle of 74° to the sun (see Figure 4). In 1840, the astronomer, John Herschel, inverted the argument and pointed out that for polarisation to occur in reflected light at right angles to the light source requires a Brewster angle of 45°, which can only occur if the medium from which the reflection occurs has a refractive index of unity. Since air has a refractive index of 1.0003, this essentially rules out reflection from anything except from a boundary between air and air itself which is nonsensical. Herschel exclaimed at the time *'the colour and polarisation of the sky are the two great standing enigmas of meteorology.'*

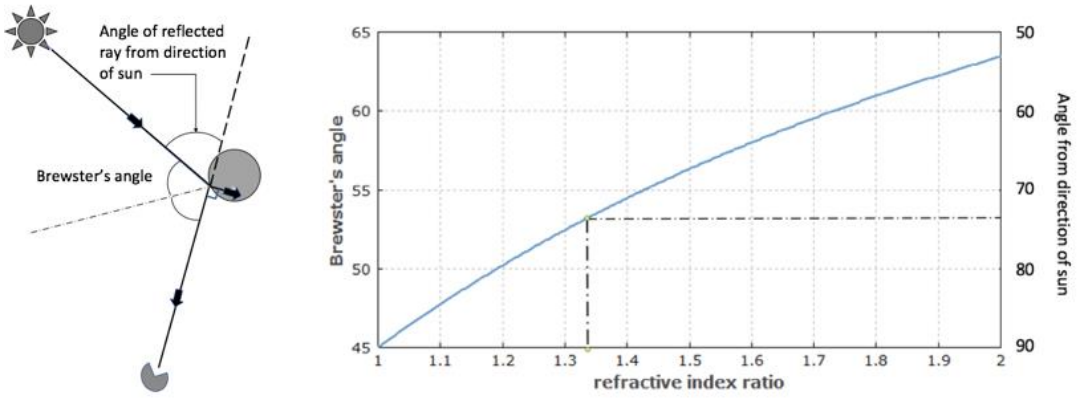


Figure 4. The ray received by external reflection of the sun's ray from a raindrop is 100% polarised when the reflected ray is at 90° to the refracted ray as shown on the left. The angle of incidence is then Brewster's angle θ_b and the angle of the reflected ray with respect to the sun's direction is $(180^o - 2\theta_b)$. In terms of refractive indices, Brewster's angle is given by $\theta_b = \tan^{-1}(n_2/n_1)$. For an air/water interface with $n_1 = 1$ and $n_2 = 1.33$, $\theta_b = 53^o$ and the angle of the reflected ray from the sun's direction is then 74°. For this angle to be 90°, θ_b would have to be 45° and hence n_1/n_2 would need to be unity as can be seen from the graph on the right. The same conclusion can be reached for reflection from the internal surface of a raindrop.

John Tyndall

The Royal Institution (RI) in London has been the laboratory of many great scientists, for example Humphry Davy, Michael Faraday and Thomas Young who has been referred to earlier. John Tyndall was another professor at the RI, an appointment that ran from 1853 to 1887. Using the apparatus shown in Figure 5, Tyndall studied the optical appearance of various vapours of liquids mixed with air when precipitated by the action of light [4].

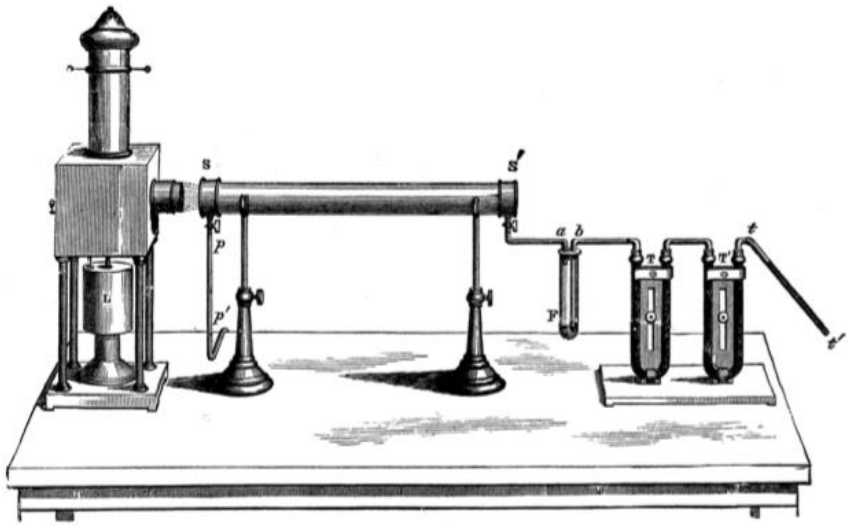


Figure 5. Apparatus used by Tyndall to study 'chemically active' clouds. Air from the right passes through filters to remove floating matter and water vapour, and then over a volatile liquid in F before entering the central observation tube SS' about one yard in length. The clouds were illuminated by the lamp L on the left. The narrow tube *pp*' was connected to a vacuum pump.

By selection of the vapours, which were, for example, carbon bisulphide, amyl nitrate, and isopropyl iodide, and, controlling their quantities, he was able obtain precipitated particles of various sizes some of which were beyond the reach of the '*highest microscopic powers*' and '*whose diameters constitute but a very small fraction of the length of a wave of violet light*'. In all such cases, and irrespective of the chemical nature of the substance, a blue cloud appeared, the colour of which in many cases *'rivalled that of the purest Italian sky'*. Of significance is the fact that the cloud was totally invisible in daylight. To be seen it needed to be surrounded by darkness and illuminated only by a powerful beam of light directed from the side. Furthermore, Tyndall found that the blue light emitted at right angles to the illuminating light beam was highly, and in some cases perfectly, polarised with the plane of polarisation also being at right angles to the beam. When the precipitated particles grew in size, the blue colouration turned to white and the emitted light was then no longer polarised.

The parallel beam of light used in these experiments tracked its way beyond the experimental tube and was visible for a length of 18 feet in the laboratory air, which Tyndall likened to *'sun-beams seen in the dusty air of London'*. The beam imparted a bluish tint to the air and when viewed laterally the light was polarised. Furthermore, observations at different angles revealed neutral points where the polarisation changed direction, just as Arago and others had observed in the sunlit sky.

After extending these investigations using tobacco smoke, condensed steam and fumes of various acids, Tyndall came to the conclusion [5] that *'the colour of the sky is due to the action of finely divided matter, rendering the atmosphere a turbid medium, through which we look at the darkness of space'*. He acknowledges that this proposal dates back to Leonardo da Vinci and Newton. In his paper, Tyndall also refers to earlier laboratory experiments using suspensions conducted by Professor M. Govi in 1860, by Rudolf Clausius, and by Ernest Brücke. However, his own experiments were the most comprehensive and convincing demonstration in the laboratory that particles of dimensions smaller than the wavelength of light reproduce the blue colour of the sky as well as its polarisation pattern. Tyndall makes the candid admission that the undulatory nature of light is unable to explain the reason for either of these related phenomena, echoing John Herschel's earlier pronouncement.

Referring to private communication with George Gabriel Stokes, Tyndall states *'I would say, if it can be demonstrated that when the particles are small in comparison to the length of a wave of light, the vibrations of a ray reflected by such particles cannot be perpendicular to the vibrations of the incident light; then assuredly the experiments recorded in the foregoing communication decide the question in favour of Fresnel's assumption.'* Fresnel, it will be recalled, had proposed that light is characterised by transverse, not longitudinal, vibrations.

The scattering of light by particulate matter is often referred to as the Tyndall effect.

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3. J.D. Hey, *From Leonardo to the Graser: Light Scattering in Historical Perspective. Part I*, South African Journal of Science 79 (1983) pp.11-27.

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Quotations in the early sections of this article have been taken either from the books referred to the In the Bibliography or from translations of the original works to be found via the internet.

1. The *Opus Maius* of Roger Bacon, translated by Robert Belle Burke, Philadelphia: University of Pennsylvania Press, 1928.
2. The reference to 'the region of fire' refers back to Aristotle's four elements making up nature, namely earth, water, air and fire, which he proposed around 340 BC to exist in concentric

spheres, in that order, around the centre of the earth, which was at that time considered to be the centre of the universe.

3. See article by Peter Tyson, *A Laboratory in the Clouds - Horace-Bénédict de Saussure (1740-1799)* in History of Physics Newsletter No.35, November 2017.
4. John Tyndall, *On the action of rays of high refrangibility upon gaseous matter*, Philosophical Transactions of the Royal Society of London 160 (1870) pp.333-365.
5. John Tyndall, *On the blue colour of the sky, the polarisation of skylight, and on the polarisation of light by cloudy matter generally*, Philosophical Magazine 37 (1869) pp.384-394.

END OF PART 1

The colour and polarisation of light from the sky: Part II

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Note: Part I of this article was published in issue 38 of the Newsletter.

Part I ended with a summary of the work of John Tyndall whose experiments at the Royal Institution on light scattered from clouds of precipitated vapours in a glass tube replicated that of the daylight sky both in its colour and the angular pattern of its polarisation. Tyndall convincingly demonstrated that both of these observations required the particles in the clouds to be small in comparison to the wavelength of the light. When he allowed the particle size to grow, the emitted light was white and unpolarised.

The investigations and findings of Tyndall in 1869-70 attracted much attention, not least that of John William Strutt, who resolved the twin problems of the sky's colour and its polarisation at age 29 just four years before he inherited, on the death of his father, the title Lord Rayleigh – alias the Third Baron Rayleigh.

1. J W Strutt: scattering from small particles

Strutt's first paper [1] on the topic begins:

'It is now, I believe, generally admitted that the light which we receive from the clear sky is due in one way or another to small suspended particles which divert the light from its regular course. On this point the experiments of Tyndall with precipitated clouds seem quite decisive. Whenever the particles of the foreign matter are sufficiently fine, the light emitted laterally is blue in colour, and, in a direction perpendicular to that of the incident beam, is completely polarized.'

Strutt states, without reservation, that the normal laws of reflection and refraction do not apply when light interacts with matter having a size less than that of the wavelength of the light. Instead, he suggests that any interaction of light with small particles is better described as *scattering*, in which the light wave is absorbed and then re-radiated in all directions. The secondary waves that are emitted at right angles to the incoming unpolarised ray must then be completely linearly polarised in a plane perpendicular to the incoming light. This is illustrated in Figure 1.

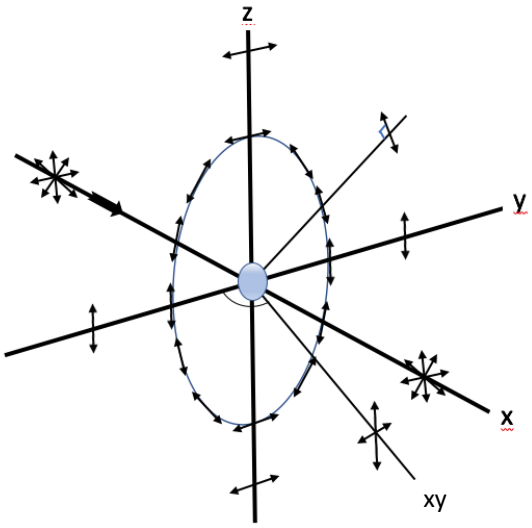

Figure 1. Unpolarised light incident from the left along the x-direction is scattered in all directions from a particle at the origin. The light scattered in the forward or backward direction along x is unpolarised, whereas light scattered in any direction in the yz plane is 100% linearly polarised. This is because the incident light, consisting of transverse vibrations, produces oscillations of the particle only in the yz plane. The oscillations that occur, for example along the z axis, can only cause re-emission along the y axis and that light must be linearly polarised in the z direction. Light scattered forwards or backwards from the yz plane is partially polarised as shown for one illustrative direction (xy) in the diagram.

Strutt recognised that the smallness of the scattering particles was not only responsible for the angle of maximum polarisation being at 90° to the incident light but also accounted for the colour of the scattered light. His theoretical treatment of the intensity of scattered light as a function of angle and wavelength of the incident light was based partly on an earlier calculation by George Gabriel Stokes [2]. However, the dependence on wavelength obtained by Strutt, namely a proportionality of the light intensity to the inverse fourth power of the wavelength, had not been obtained by Stokes. This crucial aspect of Strutt's derivation immediately accounted for the blue colour of the sky because, independently of other factors, blue light with a wavelength of 400 nm is scattered over $(700/400)^4$ = 9 times more than red light with a wavelength of 700 nm. This wavelength dependence was first derived by Strutt using dimensional analysis – see box.

> All relevant quantities can be expressed in terms of the dimension of length L. The ratio of the scattered to the incident vibrational amplitudes (A/A_0), which is clearly dimensionless, is postulated to be proportional to:
>
> (i) the volume of the particle, i.e. L^3.
> (ii) the inverse of the distance from the particle, I.e. L^{-1}.
> (iii) the wavelength to an unknown power, i.e. L^n.
>
> So, $L^3L^{-1}\ L^n$ has to be dimensionless. Therefore n = -2 and (A/A_0) must be proportional to the inverse of the wavelength squared.
>
> The corresponding ratio of intensities (I/I_0) = $(A/A_0)^2$, and hence (I/I_0) must be proportional to the inverse of the wavelength to the fourth power.

A detailed calculation gave the intensity of the light scattered at a distance r from the particle at an angle β from the direction of the incident ray as:

$$I = I_0 \frac{(\boldsymbol{D'-D})^2}{\boldsymbol{D^2}} \ (1 + \cos^2\beta) \frac{\boldsymbol{m\pi^2T^2}}{\boldsymbol{\lambda^4 r^2}}$$

where I_0 is intensity of the incident light, T is the volume of the particle and λ is the wavelength. m denotes the number of particles producing the scattering and enters simply as a multiplier because the phases of the scattered waves were considered to be random and hence the particles treated as multiple unconnected sources of light. D is the density of the medium and D' the density of the medium as altered by the particle. In this first paper, the nature of the particles performing the scattering was not specified. The only requirement was that they be small compared to the wavelength of the light and that they produce a change in density of the ether.

Note that in this paper Strutt's theory is based on the 'elastic-solid model' of the ether which, prior to Maxwell's electromagnetic theory of light (published in 1865), was the accepted model on which all theories of light were based. As we shall see, a derivation based on Maxwell's theory was published ten years later by Strutt (then Lord Rayleigh) in 1881.

The schematic below (Figure 2) illustrates roughly the angular dependence of the scattered light as predicted by the equation above. The distance of the surface of the peanut-shaped object from its centre is proportional to the intensity of the scattered light. As can be seen, the intensity in the forward direction equals that in the backward direction while the intensity of the light emitted at right angles to the incoming ray is somewhat depleted relative to that in other directions. This latter behaviour can be attributed to the radially emitted rays being restricted to light having its direction of polarisation limited to one plane, as illustrated in Figure 1.

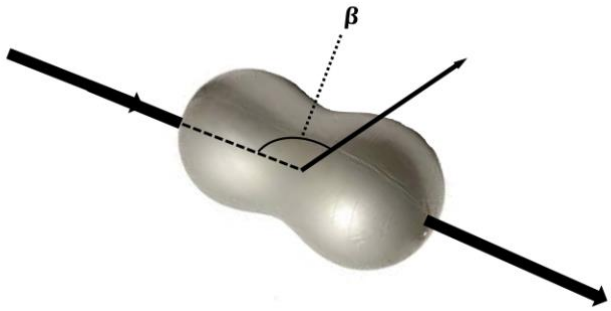

Figure 2. Schematic illustrating the angular distribution of scattered light. The surface of the 'peanut' mimics the $(1 + \cos^2\beta)$ term in Rayleigh's equation above. The reduced intensity of light scattered at right angles to the direction of the incident light is a direct consequence of the transverse nature of optical vibrations.

In order to verify his theoretical treatment, Strutt undertook experimental investigations, comparing the spectral output from the direct sun with that from the sky. Although these experimental findings (Figure 3) were never published, Strutt sketched his results on the back of an examination paper, which is extant in the Rayleigh archives. On the basis of thirty comparisons at different wavelengths, he concluded that the results were well accounted for by the inverse fourth power law.

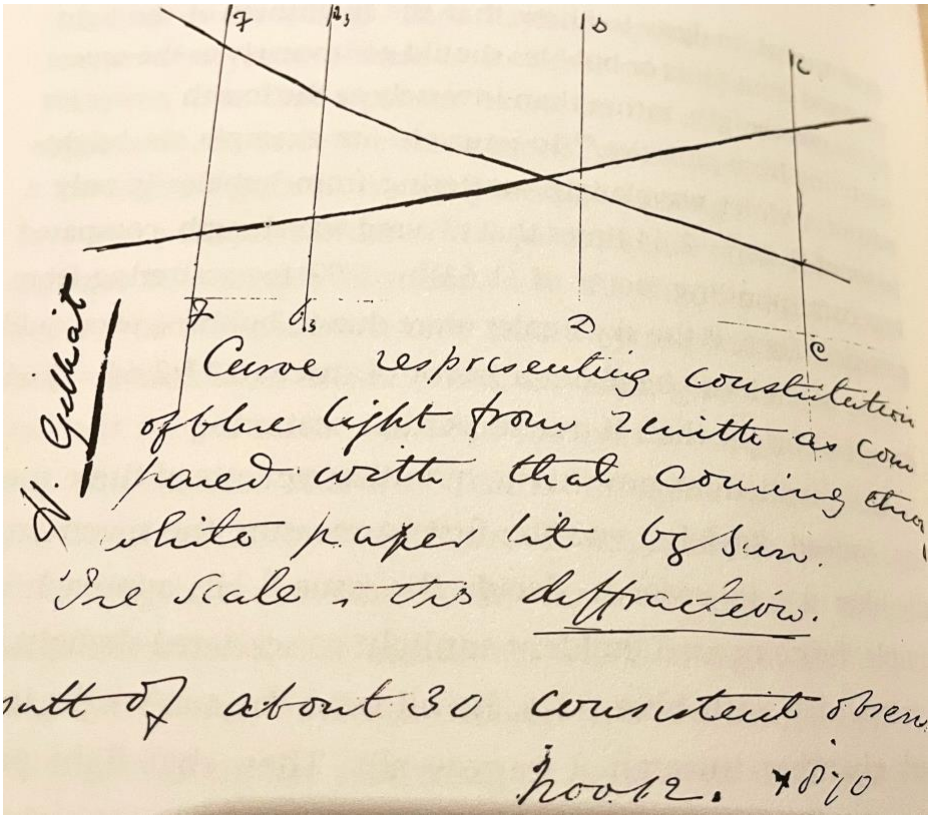

Figure 3. Unpublished experimental measurements by Strutt of the intensity of direct sunlight (curve starting at lower left) compared to that of the zenith sky (curve starting at upper left). The abscissa is wavelength, with the four vertical lines denoting the wavelengths of spectral lines associated with various elements, for example (from the left) the hydrogen blue line at 486.1 nm, the sodium yellow line at 589.3 nm, and the hydrogen red line at 656.3 nm. Clearly as the wavelength in this spectral range increases, the intensity of the scattered light falls even though that of the direct sunlight increases.

Strutt stressed that *'The principle of energy makes it clear that the light emitted is not a new creation, but only diverted from the main stream.'* Recognising that the brightness of the sky implies a significant density of scattering centres and hence that *'each particle is thus struck, not only by the direct light of the sun, but also that scattered from others'*, Strutt contended that multiple scattering could account for the measured maximum polarisation not being quite 100%.

Strutt contrasted his explanation with that proposed by many physicists from the time of Newton, namely that the light of the sky originated from the reflection of the sun's rays from thin plates of, say, ice crystals. Again, using dimensional analysis, he showed that the intensity of such reflected light varied as the inverse of the wavelength squared rather than the inverse of the wavelength to the fourth power as predicted by his scattering theory. '*There can, I think, be no question that the composition of the light of the sky agrees more nearly with the latter than with the former law*', he writes.

2. J W Strutt and J C Maxwell

In the second part [1] to this first paper, Strutt also dismisses Clausius's idea that reflection from water bubbles might be responsible. In this case he uses the argument that '*the observation that the polarisation is complete perpendicular to the track of the incident light is in itself sufficient to disprove the theory of bubbles.*' (See Figure 4 in Part I for a similar argument against invoking reflection from water drops.) This is followed by his recognition that Tyndall's work demonstrated clearly that, as the size of the scattering particle increases and eventually becomes comparable with the wavelength, the blue azure and the polarisation are both lost. He notes that this transition to a different set of phenomena will require an extension to his theory.

Strutt puts forward a very tentative suggestion as to a possible alternative to water or ice as being the scattering particles, namely common salt, but he quickly adds '*be this as it may, the optical phenomena can give us no clue*.'

The enormous advance made by Strutt in explaining both the colour and polarisation of light from the sky was acknowledged by the scientific community of the day. That someone so relatively young and comparatively unknown should have succeeded where so many before had failed, led Maxwell to exclaim '*I think Strutt on sky blue is very good*.'

Strutt's next paper on this topic [3] followed in the same year (1871) as the first. Here, he explores some approximations made in his earlier paper, mainly to ascertain if a particular observation by Tyndall could be explained if these were not made. Tyndall's observation was that when the particles in his experimental tube had grown to such a size that the light emitted normally to the incident radiation was no longer blue, viewing it through a Nicol prism '*the azure was revived in increased splendour*'. Tyndall called this 'residual blue'. Strutt conceded that his theory, which applied to exceedingly small particles, did not explain, and could not be expected to explain, this observation but he did suggest that light that had been scattered twice might have its intensity depend on wavelength to the inverse eight power and hence appear of shorter wavelength than that of the blue of the sky. No evidence of this rich blue emanating from the sky in the expected direction was found experimentally by Strutt. He suggested that that the absence of this could result from the sky being illuminated in considerable part by light from the Earth itself.

In 1873, James Clerk Maxwell wrote to Strutt as follows:

'*I have left your papers on the light of the sky, etc at Cambridge and it would take me, even if I had them, some time to get them assimilated sufficiently to answer the following question, which I think will involve less expense to the energy of the race if you stick the data into your formula and send me the results. …….. Suppose that there are N spheres of density rho and diameter s in unit volume of the medium. Find the index of refraction of the compound medium and the coefficient of extinction of light passing through it. The object of the enquiry is, of course, to obtain data about the size of molecules of air*.'

As we shall see, it was twenty-six years before the answers to these problems posed by Maxwell were answered!

3. J W Strutt inherits the title Lord Rayleigh and is appointed Cavendish Professor

In the summer of 1871, Strutt married Evelyn Balfour, the sister of Arthur Balfour who later became British Prime Minister. This required him to relinquish his fellowship at Trinity College Cambridge. At the beginning of the next year he suffered a serious attack of rheumatic fever and, as an aid to recuperation, he and his wife took a trip on the River Nile. During that voyage he wrote most of volume 1 of his classic *'Theory of Sound'*, which was eventually published in 1877. Shortly after the couple's return to England, his father died in June 1873, whereupon John William Strutt succeeded to the title of Lord Rayleigh. He was elected to Fellowship of The Royal Society in the same year. In the following years, he concentrated on building a private laboratory at the family seat in Terling, Essex and in advancing understanding of wave motion, vibrations in solids and liquids, acoustics, spectroscopy, colour vision and many other areas of physics.

Figure 4 Lord Rayleigh at his desk in Terling Place, the family seat, where he created a private laboratory.

In 1879, James Clerk Maxwell, the first holder of the post of Professor of Experimental Physics at the Cavendish Laboratory Cambridge, died and Lord Rayleigh was appointed as his successor. In the five years during which he held the Cavendish Professorship, Rayleigh published over 60 papers on a wide range of topics, including several significant ones on the determination of electrical units, and others on optics, acoustics and the behaviour of liquid jets. In 1881, in a paper entitled *'On the Electromagnetic Theory of Light'*, he returned to the matter of the scattering of light by small particles but now using Maxwell's electromagnetic theory rather than the elastic-solid ether theory on which he had based his earlier analysis [4]. Rather than the scattering centres being regions in the ether having a different density, here they are treated as regions having a different dielectric permittivity and/or magnetic permeability. The mathematics involved is quite formidable but with certain approximations Rayleigh recovered his original result that the intensity of scattering is inversely proportional to the fourth power of the wavelength. In addition, his new theory was generalised to scattering centres of arbitrary shape and size. For spherical particles beyond a certain size, consideration of higher order terms in the theory resulted in an inverse eighth power on wavelength thereby accounting for Tyndall's 'residual blue'.

In this 1881 paper, Rayleigh reports on experiments he undertook himself to confirm and extend the results found by Tyndall, namely that when the scattering particles grow in size (but still much smaller than the wavelength of the light), the direction of maximum polarisation of the scattered light is no longer perpendicular to the illuminating beam. The scattering medium that Rayleigh used was a dilute solution of hyposulphite of soda in a beaker to which had been added a small quantity of dilute

sulphuric acid. This resulted in the precipitation of sulphur, the growth of which could be arrested at any stage by the addition of ammonia. Illumination was by sunlight passing through a window into a darkened room and focussed on the beaker. Rayleigh found that the angle at which maximal polarisation occurred depended on the size of the particles as well as on the wavelength, the latter being changed with the aid of coloured glass. Of interest is the fact that the polarisation was found to be a maximum when the angle between the incoming and outgoing rays *exceeded* 90° and *increased* with the size of the particles. Intuitively, as Rayleigh notes, one might have expected the angle to have been less than 90° for larger particles in anticipation of approaching the condition for Brewster's Law to apply (see Part I of this article).

It was 1899 when Rayleigh returned to the question posed by Maxwell in 1873! His paper *'On the transmission of light though an atmosphere containing small particles in suspension, and on the origin of the blue of the sky'* [5] begins as follows:

'This subject has been treated in papers published many years ago. I resume it in order to examine more closely than hitherto the attenuation undergone by the primary light on its passage through a medium containing small particles. Closely connected with this is the interesting question whether the light from the sky can be explained by diffraction from the molecules of air themselves, or whether it is necessary to appeal to suspended particles composed of foreign matter, solid or liquid. It will appear, I think, that even in the absence of foreign particles we should still have a blue sky.'

Rayleigh emphasizes that the attenuation of sunlight by the atmosphere must be wavelength dependent, because of the scattering of the shorter-wavelength rays out of the direct path as shown by his earlier theory and as evidenced by the reddening of the sun just before sunset. To calculate the attenuation, he starts with his earlier expression for light scattered at a given angle from the main beam according to the elastic-solid theory. This is then integrated over all angles to obtain the energy lost by the primary beam. He then considers scattering from *n* particles per unit volume and deduces that the phases of the secondary (scattered) rays emitted in the same direction as the primary ray are all in phase. After some further manipulation, the final result was expressed in terms of an equation expressing the exponential fall in intensity with distance x

$$E = E_0 \exp(-hx)$$

with h given by

$$h = \frac{32\pi^3(\mu-1)^2}{3n\lambda^4}$$

where μ is the refractive index of air containing the particles.

The same result is obtained, at least for spherical bodies, on the electromagnetic theory, which Rayleigh states is '*to be preferred on every ground except that of easy intelligibility*'.

Taking λ = 6 x 10^{-13} cm (the wavelength of orange light, i.e. near the middle of the visible spectrum), and μ = 1.0003, the distance light must travel in order to undergo attenuation of 1/e = 0.37 is found to be

$$x = 4.4 \times 10^{-13} n$$

Inserting Maxwell's estimate of *n* (Loschmidt's number) = 19 x 10^{18} yields *x* = 83 kilometres

Rayleigh then writes:

'*Although Mount Everest appears fairly bright at 100 miles distance as seen from the neighbourhood of Darjeeling, we cannot suppose that the atmosphere is as transparent as is implied by the above numbers; and of course this is not to be expected, since there is certainly suspended matter to be*

reckoned with. Perhaps the best data for a comparison are those afforded by the varying brightness of stars at various altitudes.'

The decrease in brightness of a star in the zenith by passage of its light through the entire atmosphere was estimated to be about 20% according to others. Rayleigh then took this to be passage through 8.3 kilometres of air at standard pressure (Note: this estimate now seems low as about 40% of the atmosphere lies above this height, albeit of reduced density). At this rate, the transmission through 83 km would be $(0.8)^{10}$ or 0.11 of the incident radiation, compared to 0.37 calculated without allowance for any foreign matter at all. Rayleigh concludes that the light scattered from the air molecules alone are sufficient to produce a blue sky.

Reversing the argument, Rayleigh arrived at the conclusion that the number of molecules per unit volume could not be much less than estimated by Maxwell for, if it were, the transparency of the atmosphere would be less than observed. '*When we take into account the known presence of foreign matter, we shall probably see no ground for any reduction of Maxwell's number*.' (Note: today's value is 26.88×10^{18}.)

4. Gustav Mie and Ludvig Lorenz

The theory of scattering developed by Rayleigh is not valid for particles the size of which is greater than about 10% of the wavelength of the light. Clearly this limitation does not apply to sunlight scattered by the molecules of the atmosphere. Oxygen and nitrogen molecules are roughly 0.3 nm in size whereas the sun's spectrum covers wavelengths from 400 nm to 800 nm.

For larger particles, such as investigated by Tyndall in the laboratory, and, for example, water droplets in the atmosphere, the theory by the German physicist Gustav Mie, published in 1908 [6] is appropriate. In contrast to Rayleigh scattering, for which the scattering intensity is strongly wavelength dependent but identical in the forward and backward directions, Mie scattering is roughly independent of wavelength and more light is scattered forwards than backwards. See Figure 5.

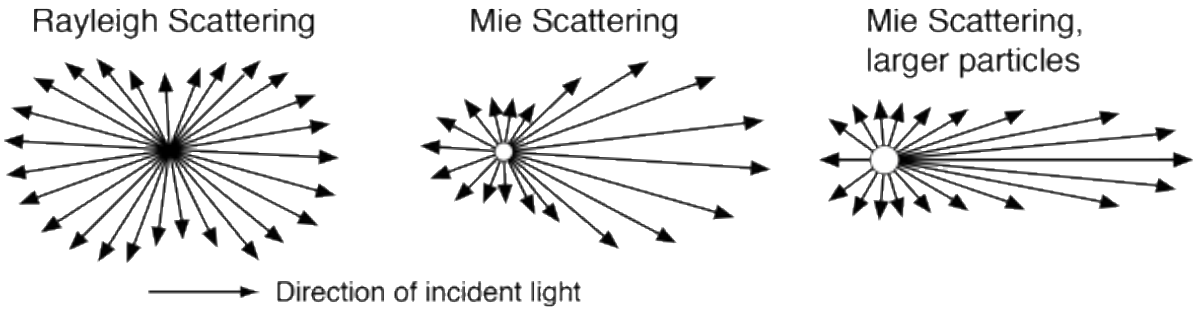

Figure 5. Spatial distribution of scattered light from particles much smaller than the wavelength of light (Rayleigh scattering) and from larger particles (Mie scattering). In the latter case, there is a predominance of light scattered in the forward direction, which increases with the particle size. Futhermore, Mie scattering depends little on the wavelength. [reproduced from http://hyperphysics.phy-astr.gsu.edu/hbase/atmos/imgatm/mie.png]

The lack of wavelength dependence of light scattering from larger particles provides a clear explanation for the white colour of clouds and the lower degree of saturation (whitening) of the sky as one approaches the horizon on account of the light having to pass through dust and other near-earth particles

Ludvig Lorenz, a Danish physicist, had in 1890 solved the same problem as Mie but, as his paper had been published in Danish in a not widely circulated journal, it remained unknown until the beginning of the 20th century [7]. In recognition of this earlier achievement, the scattering theory is sometimes

referred to as the Lorenz-Mie theory. Lorenz used the elastic-solid ether theory whereas Mie used Maxwell's EM theory.)

5. R J Strutt: Fourth Baron Rayleigh

A laboratory demonstration of light scattering from molecules of air was made by Robert John Strutt (the 4th Baron Rayleigh, son of John William Strutt) in 1918 [8]. Rather than using sunlight, which was considered impracticable in the laboratory, he used a carbon arc – the most powerful light source available at the time – to illuminate highly filtered air contained in a tube specially designed to minimise stray reflections. '*When the eyes had been rested in the dark*', Strutt observed a faint blue emission of light emitted at right angles to the incident beam. A photographic plate was used to record this scattered light and, by using coloured filters, he demonstrated that the scattered light was composed predominantly of short wavelengths, as predicted by his father's theory. Furthermore, the scattered light was found to be almost completely polarised, again as theoretically expected.

The departure of the polarisation from 100% was extensively studied by R J Strutt [9], including experiments on scattering from various gases as well as air. According to theory, complete polarisation is not to be expected for scattering from molecules that are not spherical and hence for which there are preferential directions of vibration. For air and several other gases, for example argon, nitrogen, carbon monoxide and chloroform, the polarisation was found to depart by a few percent from being complete. Hydrogen and, surprisingly, pentene produced the highest degrees of polarisation whereas helium revealed the lowest, suggesting a highly non-spherical and hence directional molecule.

A general theory of scattering from particles of arbitrary shape was given by the Third Lord Rayleigh in the same year as his son's experiments [10]. In this paper, Rayleigh stated that skylight is not expected to be perfectly polarised because of the presence of non-spherical molecules. Earlier he had given another reason: namely that light is incident on the atmosphere not only directly from the sun but also from previously scattered light and from the Earth itself.

6. Albert Einstein

An assumption made in Rayleigh's theory of scattering, and indeed in extensions of the theory by others, is that the particles involved scatter independently of each other. This ensures that the phases of the individually scattered waves are randomly distributed and that the amplitudes of the light from independent scattering events are additive. However, if the distance between individual particles producing the scattering is less than the wavelength of the light then this assumption is not valid. At low atmospheric altitudes the separation between molecules is a few nm and therefore a large number, lying within a wavelength of the exciting visible light (400-700 nm), are set into vibration simultaneously. This means that for every excited molecule there will be another for which the phase of the scattered light differs by half a wavelength, leading to net zero scattering. This mutual extinction should lead to a completely transparent atmosphere.

This apparent paradox was resolved by Einstein in 1910 [11] by noting that the number of molecules in a given volume has a statistical variation which, on average, is equal to the square root of that number. Thus, the number of molecules in a volume containing on the average N molecules will fluctuate in time within the limits $N \mp \sqrt{N}$. If A is the amplitude of the light scattered off a single molecule, the total amplitude from the non-cancelling $\sqrt{N}$ molecules is $\sqrt{N}A$ and the total intensity is NA^2. This is the same result obtained by adding the intensities from incoherent scatterers. The Rayleigh result is thus preserved, although for such a situation the scattering should be considered to arise from fluctuations in the density of the scattering centres rather than from the individual particles themselves. At high altitudes where the separation between atmospheric molecules is large, independent scattering events, as assumed in Rayleigh's theory, should still apply.

7. Envoi

Lord Rayleigh refers in his paper of 1899 to '*Mount Everest appearing fairly bright at 100 miles distance as seen from the neighbourhood of Darjeeling'* [5]. This observation illustrates the high transparency of the Earth's atmosphere, a property that is not apparent when looking skywards in daytime. Viewing vertically upwards, we gaze through about 50 miles of air, 90% of which is within 10 miles of the Earth's surface. In the daytime the scattered blue light from the sun is bright enough to make the stars and planets invisible even though the transparency of the atmosphere is about 80%.

The blue of the sky is only apparent when viewed against a black background such as that provided by empty space. In the above photograph of Mount Everest, the snow on the mountain is essentially white but the dark lower slopes are distinctly blue, matching in colour the sky above. The same is evident when viewing a less-than-full moon in daylight, the appearance of the unlit side of which is indistinguishable from the surrounding sky.

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7. See *Ludvig Lorenz, a nineteenth century theoretical physicist* by Helge Kragh (The Royal Danish Academy of Sciences and Letters 2018).
8. R.J. Strutt, *Scattering of light by dust-free air, with artificial reproduction of the blue sky – preliminary note,* Proc. Roy. Soc. A vol. 94 (1918) pp. 453–459.
9. R.J. Strutt, *The light scattered by gases; its polarisation and intensity*, Proc. Roy. Soc. A, vol. 95 (1918) pp. 155–176.
10. Lord Rayleigh, *On the scattering of light by a cloud of similar small particles of any shape and oriented at random*, Philosophical Magazine 35 (1918) pp. 373–381.
11. A. Einstein, *Theorie der Opaleszenz von homogenen Flüssigkeiten und Flüssigkeitsgemischen in der Nähe ihres kritischen Zustands,* Annalen der Physik (Leipzig) 33 (1910) pp. 1275–1298.

filtrate evaporated to dryness, gently ignited, and the mixed chlorides weighed.

The separation of potassa and soda was effected in the usual way by means of chloride of platinum.

XV. *On the Light from the Sky, its Polarization and Colour.* *By the* Hon. J. W. Strutt, *Fellow of Trinity College, Cambridge**.

IT is now, I believe, generally admitted that the light which we receive from the clear sky is due in one way or another to small suspended particles which divert the light from its regular course. On this point the experiments of Tyndall with precipitated clouds seem quite decisive. Whenever the particles of the foreign matter are sufficiently fine, the light emitted laterally is blue in colour, and, in a direction perpendicular to that of the incident beam, is *completely polarized.*

About the colour there is no *prima facie* difficulty; for as soon as the question is raised, it is seen that the standard of linear dimension, with reference to which the particles are called small, is the wave-length of light, and that a given set of particles would (on any conceivable view as to their mode of action) produce a continually increasing disturbance as we pass along the spectrum towards the more refrangible end; and there seems no reason why the colour of the compound light thus scattered laterally should not agree with that of the sky.

On the other hand, the direction of polarization (perpendicular to the path of the primary light) seems to have been felt as a difficulty. Tyndall says, "..... the polarization of the beam by the incipient cloud has thus far proved itself to be *absolutely independent of the polarizing-angle.* The law of Brewster does not apply to matter in this condition; and it rests with the undulatory theory to explain why. Whenever the precipitated particles are sufficiently fine, no matter what the substance forming the particles may be, the direction of maximum polarization is at right angles to the illuminating beam, the polarizing angle for matter in this condition being invariably 45°. This I consider to be a point of capital importance with reference to the present question" †. As to the importance there will not be two opinions; but I venture to think that the difficulty is entirely imaginary, and is caused mainly by misuse of the word reflection. Of course there is nothing in the etymology of reflection or refraction to forbid their application in this sense; but the words have acquired technical meanings, and become associated with certain well-known laws called after them. Now a moment's consideration of the principles according to which reflection and

* Communicated by the Author.

† Phil. Mag. S. 4. vol. xxxvii. p. 388.

refraction are explained in the wave theory is sufficient to show that they have no application unless the surface of the disturbing body is larger than many square wave-lengths; whereas the particles to which the sky is supposed to owe its illumination must be *smaller* than the wave-length, or else the explanation of the colour breaks down. The idea of polarization by reflection is therefore quite out of place; and that "the law of Brewster does not apply to matter in this condition" (of extreme fineness) is only what might have been inferred from the principles of the wave theory.

Nor is there any difficulty in foreseeing what, according to the wave theory, the direction of polarization ought to be. Conceive a beam of plane-polarized light to move among a number of particles, all small compared with any of the wave-lengths. The foreign matter, if optically denser than air, may be supposed to *load* the æther so as to increase its *inertia* without altering its resistance to distortion, provided that we agree to neglect effects analogous to chromatic dispersion. If the particles were away, the wave would pass on unbroken and no light would be emitted laterally. Even with the particles retarding the motion of the æther, the same will be true if, to counterbalance the increased inertia, suitable forces are caused to act on the æther at all points where the inertia is altered. These forces have the same period and *direction* as the undisturbed luminous vibrations themselves. The light actually emitted laterally is thus the same as would be caused by forces exactly the opposite of these acting on the medium otherwise free from disturbance; and it only remains to see what the effect of such forces would be.

On account of the smallness of the particles, the forces acting throughout the volume of any one are all of the same intensity and direction, and may be considered as a whole. The determination of the motion in the æther, due to the action of a periodic force at a given point, requires, of course, the aid of mathematical analysis; but very simple considerations will lead us to a conclusion on the particular point now under discussion. In the first place there is a complete symmetry round the direction of the force. The disturbance, consisting of transverse vibrations, is propagated outwards in all directions from the centre; and in consequence of the symmetry, the direction of vibration in any ray lies in the plane containing the ray and the axis; that is to say, the direction of vibration in the scattered or diffracted ray makes with the direction of vibration in the incident or primary ray the least possible angle. The symmetry also requires that the intensity of the scattered light should vanish for the ray which would be propagated along the axis; for there is nothing to distinguish one direction trans-

verse to the ray from another. We have now got what we want. Suppose, for distinctness of statement, that the primary ray is vertical, and that the plane of vibration is that of the meridian. The intensity of the light scattered by a small particle is constant, and a maximum for rays which lie in the vertical plane running east and west, *while there is no scattered ray along the north and south line.* If the primary ray is unpolarized, the light scattered north and south is entirely due to that component which vibrates east and west, and is therefore *perfectly polarized,* the direction of its vibration being also east and west. Similarly any other ray scattered horizontally is perfectly polarized, and the vibration is performed in the horizontal plane. In other directions the polarization becomes less and less complete as we approach the vertical, and in the vertical direction itself altogether disappears.

So far, then, as disturbance by very small particles is concerned, theory appears to be in complete accordance with the experiments of Tyndall and others. At the same time, if the above reasoning be valid, the question as to the direction of the vibrations in polarized light is decided in accordance with the view of Fresnel. Indeed the observation on the plane of polarization of the scattered light is virtually only another form of Professor Stokes's original test with the diffraction-grating. In its present shape, however, it is free from certain difficulties both of theory and experiment, which have led different physicists who have used the other method to contradictory conclusions. I confess I cannot see any room for doubt as to the result it leads to*.

The argument used is apparently open to a serious objection, which I ought to notice. It seems to prove too much. For if one disturbing particle is unable to send out a scattered ray in the direction of original vibration, it would appear that no combination of them (such as a small body may be supposed to be) could do so, at least at such a distance that the body subtends only a small solid angle. Now we know that when light vibrating in the plane of incidence falls on a reflecting surface at an angle of 45°, light *is* sent out according to the law of ordinary

* I only mean that *if* light, as is generally supposed, consists of transversal vibrations similar to those which take place in an elastic solid, the vibration must be normal to the plane of polarization. There is unquestionably a formal analogy between the two sets of phenomena extending over a very wide range; but it is another thing to assert that the vibrations of light are really and truly to-and-fro motions of a medium having mechanical properties (with reference to small vibrations) like those of ordinary solids. The fact that the theory of elastic solids led Green to Fresnel's formulæ for the reflection and refraction of polarized light seems amply sufficient to warrant its employment here, while the question whether the analogy is more than formal is still left open.

reflection, whose direction of vibration is perpendicular to that in the incident ray. And not only is this so in experiment, but it has been proved by Green* to be a consequence of the very same view as to the nature of the difference between media of various refrangibilities as has been adopted in this paper. The apparent contradiction, however, is easily explained. It is true that the disturbance due to a foreign body of any size is the same as would be caused by forces acting through the space it fills in a direction parallel to that in which the primary light vibrates; *but these forces must be supposed to act on the medium as it actually is—that is, with the variable density.* Only on the supposition of complete uniformity would it follow that no ray could be emitted parallel to the line in which the forces act. When, however, the sphere of disturbance is small compared with the wave-length, the want of uniformity is of little account, and cannot alter the law regulating the intensity of the vibration propagated in different directions.

Having disposed of the polarization, let us now consider how the intensity of the scattered light varies from one part of the spectrum to another, still supposing that all the particles are many times smaller than the wave-length even of violet light. The whole question admits of analytical treatment; but before entering upon that, it may be worth while to show how the principal result may be anticipated from a consideration of the *dimensions* of the quantities concerned.

The object is to compare the intensities of the incident and scattered ray; for these will clearly be proportional. The number (i) expressing the ratio of the two amplitudes is a function of the following quantities:—T, the volume of the disturbing particle; r, the distance of the point under consideration from it; λ, the wave-length; b, the velocity of propagation of light; D and D$'$, the original and altered densities: of which the first three depend only on space, the fourth on space and time, while the fifth and sixth introduce the consideration of mass. Other elements of the problem there are none, except mere numbers and angles, which do not depend on the fundamental measurements of space, time, and mass. Since the ratio i, whose expression we seek, is of no dimensions in mass, it follows at once that D and D$'$ only occur under the form D : D$'$, which is a simple number and may therefore be omitted. It remains to find how i varies with T, r, λ, b.

Now, of these quantities, b is the only one depending on time; and therefore, as i is of no dimensions in time, b cannot occur in its expression. We are left, then, with T, r, and λ; and from what we know of the dynamics of the question, we

* Camb. Phil. Trans. vol. vii.

may be sure that i varies directly as T and inversely as r, and must therefore be proportional to $T \div \lambda^2 r$, T being of three dimensions in space. In passing from one part of the spectrum to another λ is the only quantity which varies, and we have the important law:—

When light is scattered by particles which are very small compared with any of the wave-lengths, the ratio of the amplitudes of the vibrations of the scattered and incident light varies inversely as the square of the wave-length, and the intensity of the lights themselves as the inverse fourth power.

I will now investigate the mathematical expression for the disturbance propagated in any direction from a small particle which a beam of light strikes.

Let the vibration corresponding to the incident light be expressed by $A \cos \frac{2\pi}{\lambda} bt$. The acceleration is

$$-A\left(\frac{2\pi}{\lambda} b\right)^2 \cos \frac{2\pi}{\lambda} bt;$$

so that the force which would have to be applied to the parts where the density is D′, in order that the wave might pass on undisturbed, is, per unit of volume,

$$-(D'-D)A\left(\frac{2\pi b}{\lambda}\right)^2 \cos \frac{2\pi}{\lambda} bt.$$

To obtain the total force which must be supposed to act over the space occupied by the particle, the factor T must be introduced. The opposite of this conceived to act at O (the position of the particle) gives the same disturbance in the medium as is actually caused by the presence of the particle. Suppose, now, that the ray is incident along O Y, and that the direction of vibration makes an angle α with the axis of x, which is the line of the scattered ray under consideration—a supposition which involves no loss of generality, because of the symmetry which we have shown to exist round the line of action of the force. The question is now entirely reduced to the discovery of the disturbance produced in the æther by a given periodic force acting at a fixed point in it. In his valuable paper "On the Dynamical Theory of Diffraction"*, Professor Stokes has given a complete investigation of this problem; and I might assume the result at once. The method there used is, however, for this particular purpose very indirect, and accordingly I have thought it advisable to give a comparatively short cut to the result, which will be found at the end of the present paper. It is proved that if the total force acting at O in the manner supposed be $F \cos \frac{2\pi}{\lambda} bt$, the re-

* Camb. Phil. Trans. vol. ix.

sulting disturbance in the ray propagated along O X is

$$\zeta = \frac{\mathrm{F}\sin\alpha}{4\pi b^2 \mathrm{D}r}\cos\frac{2\pi}{\lambda}(bt-r).$$

Substituting for F its value, we have

$$\zeta = \mathrm{A}\frac{\mathrm{D}'-\mathrm{D}}{\mathrm{D}}\frac{\mathrm{T}}{r\lambda^2}\sin\alpha\cos\frac{2\pi}{\lambda}(bt-r),$$

an equation which includes all our previous results and more.

One reservation, however, must not be omitted. Since we have supposed the medium uniform throughout, whereas it really has a different density at the place where the force acts, our investigation does not absolutely correspond to the actual circumstances of the case. As before remarked, no error is on that account to be feared in the law determining the intensity of the vibration in different directions; but it is probable that the coefficient, so far as it depends on D : D′, may be changed*, and there may be a change in the phase comparable with $\frac{2\pi}{\lambda}\times$ the linear dimension of the particle, which is of importance when the scattered and primary waves have to be compounded.

So much for a single particle. In actual experiments, as, for instance, with Professor Tyndall's "clouds," we have to deal with an immense number of such particles; and the question now is to deduce what their effect must be from the results already obtained. Were the particles absolutely motionless, the partial waves sent out in any direction from them would have permanent relations as to phase, and the total disturbance would have to be found by compounding the *vibrations* due to all the particles. Such a supposition, however, would be very wide of the mark; for, in consequence of the extreme smallness of λ, the slightest motion of any particle will cause an alteration of phase passing through many periods in a less time than the eye could appreciate. Our particles are, then, to be treated as so many *unconnected* sources of light; and instead of adding the *vibrations*, we must take the *intensities* represented by their squares. Only in one direction is a different treatment necessary, namely along the course of the primary light. I mention this because it would not otherwise appear how the reduction in the intensity of the transmitted light is effected; but we do not require to follow the details of the process, because, when once we know the intensity of the light emitted laterally, the principle of energy will tell us what the primary wave has lost.

* I find that no alteration of any kind is needed.—Jan. 20.

The intensity of the light scattered from a cloud is thus equal to

$$A^2 \frac{(D'-D)^2}{D^2} \sin^2\alpha \frac{\Sigma T^2}{\lambda^4 r^2},$$

where ΣT^2 is the sum of all the squares of T. If T^2 be understood to denote the mean square of T (*not* the square of the mean value of T), and m be the number of particles,

$$\Sigma T^2 = m \,.\, T^2.$$

If the primary light be unpolarized, the intensity in a direction making an angle β with its course becomes

$$A^2 \frac{(D'-D)^2}{D^2} (1+\cos^2\beta) \frac{mT^2}{\lambda^4 r^2}.$$

Backwards from the cloud the light is thus twice as bright as normally. To the light scattered nearly in the direction of the primary ray our expression does not apply.

Fig. 1.

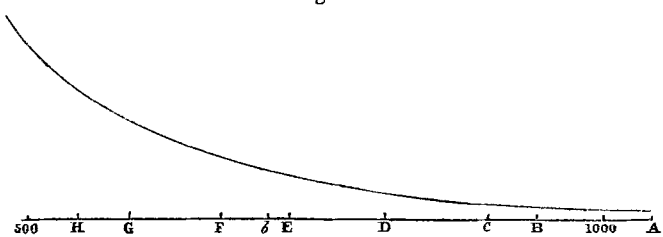


Fig. 1 shows the curve representing the intensity of the scattered light for each part of the spectrum, referred to the intensity in the primary light as a standard. The abscissa being proportional to λ, the base line represents the diffraction-spectrum with the principal fixed lines. Over the brighter portion of the spectrum from B to G the curve differs but little from a straight line, while the small curvature is turned downwards, indicating a deficiency in the green and yellow.

Before making out the theory, I had endeavoured to ascertain by observation the actual prismatic composition of the blue of the sky, and had obtained preliminary results. The experimental method (the description of which I must reserve for another opportunity) was fully adequate to the comparison of two given lights; but the difficulty was to find something to compare the blue light with. In the only complete set of observations that I have hitherto been able to make, the blue of the sky (apparently a very good one) taken from the neighbourhood of the zenith was compared with sunlight diffused through white paper.

About thirty consistent comparisons were made, ranging over the spectrum from C to beyond F, and a curve drawn on the plan of fig. 1. I do not give the complete curve, because I hope before long to complete and confirm the observations; but the following numbers will give an idea of the result:—

C.	D.	b_3.	F.	
25	40	63	80	from fig. 1.
25	41	71	90	observed.

The upper line gives the theoretical intensities for the fixed lines C, D, b_3, F, while the lower gives the observed ratios between the lights (sky and diffused sunlight), the two sets of numbers being made to agree at C. Considering the difficulties and uncertainties of the case, the two curves agree very well; and it should be noticed that the sky compared with diffused light was even bluer than theory makes it, on the supposition that the diffused light through the paper may be taken as similar to that whose scattering illuminates the sky. It is possible that the paper was slightly yellow; or the cause may lie in the yellowness of sunlight as it reaches us compared with the colour it possesses in the upper regions of the atmosphere. It would be a mistake to lay any great stress on the observations in their present incomplete form; but at any rate they show that a colour more or less like that of the sky would result from taking the elements of white light in quantities proportional to λ^{-4}. I do not know how it may strike others; but individually I was not prepared for so great a difference as the observations show, the ratio for F being more than three times as great as for C.

There is one point in which our calculations do not exactly meet the case of the sky. In the experiments with precipitated clouds the total quantity of light scattered is quite insignificant compared with the incident beam; but it is by no means so clear that the same is the case with the sky. Each particle is thus struck, not only by the direct light of the sun, but also by that scattered from others. It does not seem that the chromatic effects would be much affected by this consideration; but it is worth notice that the conclusion as to complete polarization perpendicular to the incident ray would have to be modified. To see this, imagine, as before, the light (unpolarized) incident along O Y upon a particle O; we have seen that the ray diffracted along O X contains no vibration parallel to O Y. By the aid, however, of another particle P in the *xy* plane such a vibration may be communicated to it; for in the ray diffracted from P to O there is a component vibration in the *xy* plane perpendicular to P O, which, when again diffracted along O X, will give a component parallel to O Y. This is perhaps the explanation

of the incomplete polarization of sky-light at right angles to the solar beams; but it must be remembered that an insufficient fineness in some of the particles of foreign matter would have a like result.

By many physicists, from Newton downwards, the light of the sky has been supposed to be reflected from thin plates, and the colour to be the blue of the first order in Newton's scale. Such a view is fundamentally different from that adopted in this paper, though it might not at first seem so. In support of this assertion, it may be sufficient to notice that the two theories are at variance as to the law connecting the intensity with wave-length. By an argument from dimensions similar to that already used, it is easy to find how the intensity of the light reflected from a thin plate (thin, that is, compared with *any* of the wave-lengths) varies with λ. Instead of our former quantities, T, r, λ, we now have merely λ, and δ the thickness of the plate. Since the reflected vibration necessarily varies as δ, it must also be proportional to λ^{-1}, and so the *intensity* of the reflected light $\propto\lambda^{-2}$ instead of λ^{-4}. The ordinary analytical expression for the reflected light leads readily to the same conclusion (Airy's Tracts, p. 297). There can, I think, be no question that the composition of the light of the sky agrees more nearly with the latter than with the former law.

The principle of energy makes it clear that the light emitted laterally is not a new creation, but only diverted from the main stream. If I represent the intensity of the primary light after traversing a thickness x of the turbid medium, we have

$$dI = -kI\lambda^{-4}dx,$$

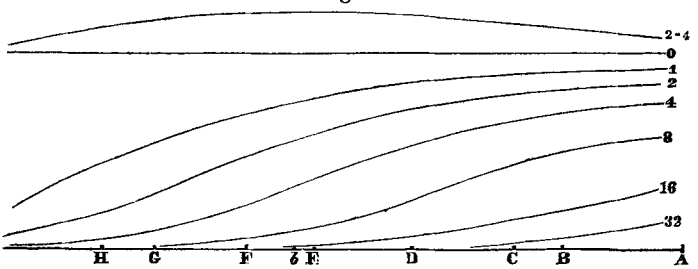
where k is a constant independent of λ. On integration,

$$I = I_0 e^{-k\lambda^{-4}x},$$

if I_0 correspond to $x=0$,—a law altogether similar to that of absorption, and showing how the light tends to become yellow and finally red as the thickness of the medium increases. Fig 2 shows a series of curves representing the composition of the originally white light after passing through thicknesses in the ratio of 1, 2, 4, 8, 16, 32. The reader will observe how little of the violet light remains when the red is still in nearly its original force. I cannot but think that this rapid diversion of the rays of short wave-length has a good deal to do with the absence of light of the highest refrangibility from the direct rays of the sun. For the line A at the extreme red and R near the upper limit of the photographic spectrum the wave-lengths are 7617 and 3108. The ratio of the fourth powers is about 36 : 1; so that, whatever

the fraction representing the transmission of A may be, its 36th power will give the transmission of R. To take an instance, if

Fig. 2.



·9 of the ray A gets through, only ·018 of R would be able to penetrate. For the rays of still higher refrangibility, which Professor Stokes found abundant in the electric light but missing in the solar rays, the fraction would be smaller still; but I am not aware of any measurements of smaller wave-length on which to found a calculation.

We have hitherto supposed that the light scattered by the finely divided matter reaches the eye without modification, and we have taken no account of any change in the composition of the primary light before diffraction. If x be the total length of the path of the ray through the turbid medium, we may express the quality of the light in terms of x; for it makes no difference whether the lateral leakage takes place before diffraction or after. In fact

$$I \propto \lambda^{-4} \epsilon^{-k\lambda^{-4}x},$$

an expression which shows that I vanishes for very small as well as for very large values of λ, while for some definite value (say Λ) it rises to a maximum (I_0). Expressing I in terms of I_0 and Λ, we have

$$\frac{I}{I_0} = \frac{\Lambda^4}{\lambda^4} \epsilon^{1-\frac{\Lambda^4}{\lambda^4}};$$

from which we may fall back on our original law by supposing Λ indefinitely small, and replacing $\Lambda^4 I_0$ by a finite constant. An approximate idea of the character of these lights may be obtained by subtracting the successive curves of fig. 2. Thus the difference of the curves marked 2 and 4 represents a light having its maximum brightness (of course relatively to the primary light) in the blue-green portion of the spectrum. I find by calculation that, if the maximum intensity be at b and be taken as unity, the intensities at G and C are given by the numbers ·713, ·710 respectively. The colour would be greenish; but whether

the green of the sky is to be accounted for in this way I am not able to say. Some, I believe, consider it to be entirely a contrast effect.

Appendix.

Within a space T, small in all its dimensions against λ, and situated at the origin of coordinates, let a force parallel to O Z, and, so far as it depends upon the time, expressed by a simple circular function, act on the medium. If ξ, η, ζ denote the displacements parallel to the axes at the point xyz, and

$$\delta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz},$$

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= b^2\nabla\xi + (a^2-b^2)\frac{d\delta}{dx}, \\ \frac{d^2\eta}{dt^2} &= b^2\nabla\eta + (a^2-b^2)\frac{d\delta}{dy}, \\ \frac{d^2\zeta}{dt^2} &= b^2\nabla\zeta + (a^2-b^2)\frac{d\delta}{dz} + \mathrm{Z}, \end{aligned}\right\} \quad . \quad . \quad (\mathrm{A})^*$$

where ∇ stands for $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$; a^2 and b^2 are constants depending on the nature of the medium supposed to be isotropic. For the luminiferous æther, Green has shown that a is to be regarded as indefinitely great†.

To represent the periodic force, write for Z, $\mathrm{Z}\epsilon^{int}$. Similar transformations will then apply to ξ, η, ζ, and δ; so that on substitution in (A) and dividing out the common factor ϵ^{int}, there results

$$\left.\begin{aligned} (b^2\nabla + n^2)\xi + (a^2-b^2)\frac{d\delta}{dx} &= 0, \\ (b^2\nabla + n^2)\eta + (a^2-b^2)\frac{d\delta}{dy} &= 0, \\ (b^2\nabla + n^2)\zeta + (a^2-b^2)\frac{d\delta}{dz} &= -\mathrm{Z}. \end{aligned}\right\} \quad . \quad . \quad (\mathrm{B})$$

Writing

$$\frac{d\xi}{dy} - \frac{d\eta}{dx} = \varpi_3 \text{ \&c.},$$

* Thomson and Tait's 'Natural Philosophy,' p. 530.

† Camb. Phil. Trans vol. vii

we obtain from (B) by differentiation and subtraction,

$$\left.\begin{aligned}(b^2\nabla + n^2)\varpi_3 &= 0,\\ (b^2\nabla + n^2)\varpi_1 &= \frac{dZ}{dy},\\ (b^2\nabla + n^2)\varpi_2 &= -\frac{dZ}{dx}.\end{aligned}\right\} \quad \ldots\ldots \quad (C)$$

ϖ_1, ϖ_2, ϖ_3 are the *rotations* of the elements of the medium round axes parallel to those of coordinates.

The disturbance which we are investigating is that caused and maintained by the force Z acting within the space T. Accordingly*

$$\varpi_3 = 0,$$
$$\varpi_1 = -\frac{1}{4\pi b^2}\iiint \frac{dZ}{dy}\frac{\epsilon^{\pm ikr}}{r}\,dx\,dy\,dz,$$

r being the distance between the element $dx\,dy\,dz$ and the point where ϖ_1 is estimated, and

$$k = \frac{2\pi}{\lambda} = \frac{n}{b}. \quad \ldots\ldots\ldots \quad (D)$$

Since $\epsilon^{\pm ikr}$ will be finally multiplied by ϵ^{int}, and the disturbance which we are dealing with is propagated *outwards* from T, it is evident that the *lower* sign is to be employed. Now

$$\int \frac{\epsilon^{-ikr}}{r}\frac{dZ}{dy}\,dy = \left[Z\frac{\epsilon^{-ikr}}{r}\right] - \int Z\frac{d}{dy}\left(\frac{\epsilon^{-ikr}}{r}\right)dy,$$

of which the term within brackets vanishes, because the value of Z is only finite within the space T. Thus

$$\varpi_1 = \frac{1}{4\pi b^2}\iiint Z\frac{d}{dy}\left(\frac{\epsilon^{-ikr}}{r}\right)dx\,dy\,dz.$$

The factor $\frac{d}{dy}\left(\frac{\epsilon^{-ikr}}{r}\right)$ within the space T is sensibly constant, so that, if Z stand for the mean value of Z over the volume T,

$$\varpi_1 = \frac{TZ}{4\pi b^2}\frac{d}{dy}\left(\frac{\epsilon^{-ikr}}{r}\right),$$
$$\varpi_2 = \frac{-TZ}{4\pi b^2}\frac{d}{dx}\left(\frac{\epsilon^{-ikr}}{r}\right),$$
$$x\varpi_1 + y\varpi_2 \propto \left(x\frac{d}{dy} - y\frac{d}{dx}\right)\cdot\frac{\epsilon^{-ikr}}{r} = 0.$$

* Helmholtz, Crelle's *Journal*, 1860.

And if $R = \sqrt{x^2+y^2}$,

$$\varpi = \frac{x\varpi_2 - y\varpi_1}{R} = \frac{-TZ}{4\pi b^2 R}\left(x\frac{d}{dx}+y\frac{d}{dy}\right)\cdot\frac{\epsilon^{-ikr}}{r}$$

$$= \frac{-TZ}{4\pi b^2}\frac{d}{dR}\cdot\frac{\epsilon^{-ikr}}{r} = \frac{TZ\sin\alpha}{4\pi b^2}\frac{d}{dr}\cdot\frac{\epsilon^{-ikr}}{r},$$

where α denotes the angle between r and z.

The resultant rotation at any point is thus about an axis perpendicular to the plane passing through the point and the axis of Z; and its magnitude is given by ϖ. In differentiating $\frac{\epsilon^{-ikr}}{r}$ with respect to r, we may neglect the term divided by r^2 as altogether insensible, kr being an exceedingly great quantity at any ordinary distance from the origin of disturbance. Thus

$$\varpi = \frac{-ik\,.\,TZ\sin\alpha}{4\pi b^2}\cdot\frac{\epsilon^{-ikr}}{r}, \quad . \quad . \quad . \quad . \quad . \quad (E)$$

which completely determines the rotation at any point. For a given disturbance it is seen to be everywhere about an axis perpendicular to r and the direction of the force, and in magnitude dependent only on the angle between these two directions (α) and on the distance (r).

The intensity of the light, however, is more usually expressed in terms of the actual displacement in the plane of the wave. In order to find the connexion between the two quantities, it will be more convenient to suppose the scattered ray parallel to x, and that the force F (for Z is no longer appropriate) acts in the plane of zx at an angle α with Ox. ϖ becomes identical with ϖ_2; that is, with $\frac{d\zeta}{dx}$; for ξ as well as η is zero; so that

$$\zeta = \int \varpi dr = \frac{TF\sin\alpha}{4\pi b^2}\cdot\frac{\epsilon^{-ikr}}{r}.$$

Restoring the factor ϵ^{int}, we have

$$\zeta = \frac{TF\sin\alpha}{4\pi b^2}\cdot\frac{\epsilon^{i(nt-kr)}}{r};$$

or throwing away the imaginary part,

$$\zeta = \frac{TF\sin\alpha}{4\pi b^2 r}\cdot\cos\frac{2\pi}{\lambda}(bt-r). \quad . \quad . \quad . \quad . \quad (F)$$

This corresponds to a total accelerating force equal to

$$\mathrm{FT}\cos\frac{2\pi}{\lambda}bt;$$

a result which agrees with that of Professor Stokes's more complete investigation, with the exception of a slight difference of notation.

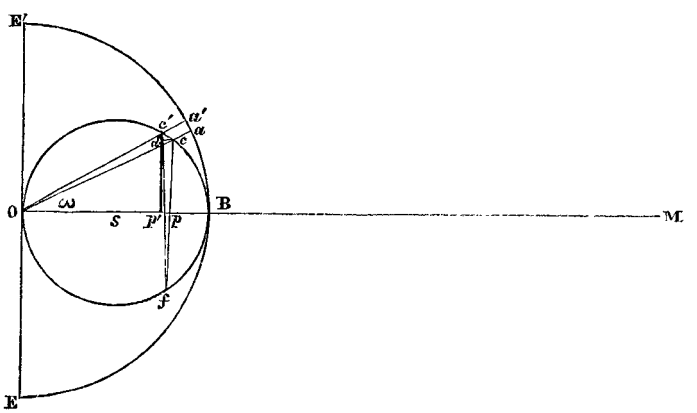
Terling Place, Witham,
December 30, 1870.

[To be continued.]

XVI. *Addendum to a Paper on the Theory of the Tides. By the* Rev. T. K. Abbott, *Fellow and Tutor of Trinity College, Dublin*.*

IN a former Number of this Magazine (Jan. 1870) I gave elementary proofs of the principal theorems in the Theory of the Tides. In order to complete the elementary treatment of the subject, it is desirable to give simple constructions for the disturbing force, the velocity of the current, and the height of the tide. The two latter I propose to give in the present paper; it would be superfluous to occupy your space with the former.

It is in fact readily shown by elementary geometry that in the figure †, where E B E′ represents the hemisphere under the



moon, and a circle O c' c B f is described round the radius directed to the moon, the tangential disturbing force at a is proportional to the perpendicular cp. If aa' be the space passed over in the

* Communicated by the Author.

† By inadvertence the direction of the motion in this figure is the reverse of that in the figure in the former paper.

Water at 18°·9 C. gave $u = 11° 35'$, $d = 0{\cdot}9996$
„ 69°·0 „ $u_1 = 10° 35'$, $d' = 0{\cdot}9799$

when the tangent-balance was adjusted to the density of absolute alcohol 0·7946.

From this are obtained the volumes

$$\begin{aligned} v &= 1{\cdot}000400 \\ v_1 &= 1{\cdot}002051 \\ \hline \Delta v &= 0{\cdot}001651 \end{aligned}$$

for $\Delta t = 50°{\cdot}1$ C.

The correction for the expansion of the glass rod is $d\phi = 0{\cdot}00451$, and hence the actual expansion

$$\Delta v = 0{\cdot}002102.$$

According to Despretz, the volume of water at

$$\begin{aligned} 18°{\cdot}9\ \text{C.}\quad v &= 1{\cdot}00156 \\ 69°{\cdot}0\quad v_1 &= 1{\cdot}02200 \\ \hline dv &= 0{\cdot}02044 \end{aligned}$$

Prague, December 17, 1870.

LVIII. *On the Scattering of Light by small Particles. By the* Hon. J. W. Strutt, *Fellow of Trinity College, Cambridge**.

THE investigation of the diffraction of light by small particles, contained in the February Number of this Magazine, proceeds throughout on the assumption that the difference between two media which differ in refractive power is a difference of density and not a difference of rigidity. My object in the present communication is to attack the problem more generally, and to show that the more special hypothesis is in no degree arbitrary, but forced upon us by the phenomena themselves. The words "density," "rigidity" need not be interpreted literally, but are used in a generalized sense analogous to that given to "velocity" and "force" in the higher mechanics.

The first step is to find the equation of motion of an isotropic elastic medium whose density and rigidity may vary from point to point. If D denote the density and n the rigidity, a process similar to that used in Thomson and Tait's 'Natural Philosophy,' p. 530, leads us to the following:—

$$\begin{aligned} &\frac{d}{dx}(m\delta) + \nabla_n \xi - \mathrm{D}\frac{d^2\xi}{dt^2} \\ &\quad - \frac{dn}{dx}\frac{d\eta}{dy} - \frac{dn}{dx}\frac{d\zeta}{dz} + \frac{dn}{dy}\frac{d\eta}{dx} + \frac{dn}{dz}\frac{d\zeta}{dx} = 0, \quad . \quad . \quad (1) \end{aligned}$$

* Communicated by the Author.

and two similar equations, where ξ, η, ζ are the displacements parallel to the coordinate axes,

$$\delta \equiv \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz},$$

$$\nabla_n \equiv \frac{d}{dx}\left(n\frac{d}{dx}\right) + \frac{d}{dy}\left(n\frac{d}{dy}\right) + \frac{d}{dz}\left(n\frac{d}{dz}\right).$$

If n and D were constant, equations (1) would be satisfied by

$$\left.\begin{aligned} \xi = \eta = \delta = 0, \\ \zeta = \zeta_0 = \epsilon^{ik(bt+x)}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

In the application that we have to make, n and D may be supposed to be constant except within a small space T at the origin of coordinates, where they assume the values $n+\Delta n$, $D+\Delta D$. In consequence of this variation the equations of motion are no longer satisfied by (2); but we may take as the true values of the displacements, ξ, η, $\zeta_0+\zeta$, δ, where ξ, η, ζ, δ are small quantities of the order T, which are to be neglected when multiplied by Δn, ΔD*. Substituting in equations (1) and dropping the factor relating to the time, we get

$$\left.\begin{aligned} \frac{d}{dx}(m\delta) + \nabla_n \xi + Dk^2b^2\xi + \frac{dn}{dz}\frac{d\zeta_0}{dx} &= 0, \\ \frac{d}{dy}(m\delta) + \nabla_n \eta + Dk^2b^2\eta &= 0, \\ \frac{d}{dz}(m\delta) + \nabla_n \zeta + Dk^2b^2\zeta + \nabla_n \zeta_0 + Dk^2b^2\zeta_0 &= 0, \end{aligned}\right\} \quad (3)$$

and

$$k = \frac{2\pi}{\lambda}, \quad b^2 = \frac{n}{D},$$

or, since

$$\nabla_n \zeta_0 = \nabla_{\Delta n}\zeta_0 + \nabla_n \zeta_0,$$

$$D\zeta_0 = (D+\Delta D)\zeta_0 = D\zeta_0 + \Delta D \,.\, \zeta_0,$$

and

$$\nabla_n \zeta_0 + nk^2\zeta_0 = 0,$$

$$\left.\begin{aligned} \frac{1}{n}\frac{d}{dx}(m\delta) + \nabla\xi + k^2\xi + \frac{1}{n}\frac{dn}{dz}\frac{d\zeta_0}{dx} &= 0, \\ \frac{1}{n}\frac{d}{dy}(m\delta) + \nabla\eta + k^2\eta &= 0, \\ \frac{1}{n}\frac{d}{dz}(m\delta) + \nabla\zeta + k^2\zeta + \frac{1}{n}\nabla_{\Delta n}\zeta_0 + \frac{\Delta D}{D}k^2\zeta_0 &= 0. \end{aligned}\right\}$$

Hence if $\varpi_3 = \frac{d\xi}{dy} - \frac{d\eta}{dx}$ &c. be the *rotations* of the medium,

* The effect of this is subsequently considered.

$$\left.\begin{aligned}
\nabla\varpi_3+k^2\varpi_3+\frac{1}{n}\frac{d}{dy}\left(\frac{dn}{dz}\frac{d\zeta_0}{dx}\right) &= 0,\\
\nabla\varpi_1+k^2\varpi_1-\frac{d}{dy}\left\{\frac{1}{n}\frac{d}{dx}\left(\Delta n\frac{d\zeta_0}{dx}\right)+\frac{\Delta D}{D}k^2\zeta_0\right\} &= 0,\\
\nabla\varpi_2+k^2\varpi_2+\frac{d}{dx}\left\{\frac{1}{n}\frac{d}{dx}\left(\Delta n\frac{d\zeta_0}{dx}\right)+\frac{\Delta D}{D}k^2\zeta_0\right\}& \\
-\frac{1}{n}\frac{d}{dz}\left\{\frac{dn}{dz}\frac{d\zeta_0}{dz}\right\} &= 0.
\end{aligned}\right\} \quad . \quad (4)$$

Accordingly

$$\begin{aligned}
\varpi_3 &= \frac{1}{4\pi n}\iiint\frac{\epsilon^{-ikr}}{r}\frac{d\zeta_0}{dx}\frac{d}{dy}\left(\frac{d\Delta n}{dz}\right)dv,\\
&= -\frac{1}{4\pi n}\iiint\frac{d\Delta n}{dz}\frac{d\zeta_0}{dx}\frac{d}{dy}\left(\frac{\epsilon^{-ikr}}{r}\right)dv,\\
&= -\frac{1}{4\pi n}\iiint\Delta n\frac{d\zeta_0}{dx}\frac{d^2}{dy\,dz}\left(\frac{\epsilon^{-ikr}}{r}\right)dv,\\
&= \frac{\Delta n\,T}{4\pi n}\frac{d\zeta_0}{dx}\cdot\frac{yz}{r^2}\frac{d^2}{dr^2}\left(\frac{\epsilon^{-ikr}}{r}\right),
\end{aligned}$$

higher powers of $\frac{1}{r}$ being neglected. By similar reasoning,

$$\begin{aligned}
\varpi_1 &= \frac{-T}{4\pi}\frac{\Delta D}{D}k^2\zeta_0\frac{y}{r}\frac{d}{dr}\left(\frac{\epsilon^{-ikr}}{r}\right)\\
&\quad -\frac{T}{4\pi}\frac{\Delta n}{n}\frac{d\zeta_0}{dx}\frac{xy}{r^2}\frac{d^2}{dr^2}\left(\frac{\epsilon^{-ikr}}{r}\right),\\
\varpi_2 &= \frac{T}{4\pi}\frac{\Delta D}{D}k^2\zeta_0\frac{x}{r}\frac{d}{dr}\left(\frac{\epsilon^{-ikr}}{r}\right)\\
&\quad +\frac{T}{4\pi}\frac{\Delta n}{n}\frac{d\zeta_0}{dx}\frac{x^2}{r^2}\frac{d^2}{dr^2}\left(\frac{\epsilon^{-ikr}}{r}\right)\\
&\quad -\frac{T}{4\pi}\frac{\Delta n}{n}\frac{d\zeta_0}{dx}\frac{z^2}{r^2}\frac{d^2}{dr^2}\left(\frac{\epsilon^{-ikr}}{r}\right);
\end{aligned}$$

or, since

$$\zeta_0=\epsilon^{ikx},\quad \frac{d\zeta_0}{dx}=ik\epsilon^{ikx},$$

$$\left.\begin{aligned}
\varpi_3 &= \frac{ik^3T}{4\pi}\frac{\epsilon^{-ikr}}{r}\left(-\frac{\Delta n}{n}\frac{yz}{r^2}\right),\\
\varpi_1 &= \frac{ik^3T}{4\pi}\frac{\epsilon^{-ikr}}{r}\left(\frac{\Delta D}{D}\frac{y}{r}+\frac{\Delta n}{n}\frac{xy}{r^2}\right),\\
\varpi_2 &= \frac{ik^3T}{4\pi}\frac{\epsilon^{-ikr}}{r}\left(-\frac{\Delta D}{D}\frac{x}{r}+\frac{\Delta n}{n}\frac{z^2-x^2}{r^2}\right).
\end{aligned}\right\} \quad . \quad (5)$$

These are the component rotations. The resultant in the general case would be rather complicated, and is not wanted for our purpose. It is easily seen to be about an axis perpendicular to the scattered ray, inasmuch as

$$x\varpi_1+y\varpi_2+z\varpi_3=0.$$

Let us consider the particular case of a ray scattered normally to the incident light, so that $x=0$. Denoting for brevity the common factor by p, we have

$$\left.\begin{aligned}\varpi_3&=-p\frac{\Delta n}{n}\frac{yz}{r^2},\\ \varpi_1&=\ p\frac{\Delta \mathrm{D}}{\mathrm{D}}\frac{y}{r},\\ \varpi_2&=\ p\frac{\Delta n}{n}\frac{z^2}{r^2},\end{aligned}\right\}\quad\ldots\ldots\quad(6)$$

whence

$$\varpi^2=\varpi_1^2+\varpi_2^2+\varpi_3^2=p^2\left(\frac{\Delta n}{n}\right)^2\frac{z^2}{r^2}+p^2\left(\frac{\Delta \mathrm{D}}{\mathrm{D}}\right)^2\frac{y^2}{r^2}.$$

Here we have reached a result of some importance and one which can be confronted with fact. For from the value of ϖ it appears that there is no direction in the plane perpendicular to an incident ray of polarized light in which the scattered light vanishes, if Δn and $\Delta \mathrm{D}$ be both finite. Now experiment tells us plainly that there is such a direction, and therefore we may infer with certainty that either Δn *or* $\Delta \mathrm{D}$ vanishes. So far we have a choice between two suppositions; either we may assume, as in my former paper, that there is no difference of *rigidity* between one medium and another, and that the vibrations of light are normal to the plane of polarization, or, on the other hand, that there is no difference of *density* between media, and then the vibrations must be supposed to be *in* the plane of polarization. The former view is the one adopted by Green and (virtually) by Cauchy in their theories of reflection; while the latter is that of MacCullagh and Neumann, which I now proceed to show is untenable.

Suppose then that $\Delta \mathrm{D}=0$. Reverting to the general values of ϖ_1, ϖ_2, ϖ_3 in (5), we have

$$\left.\begin{aligned}\varpi_3&=-p\frac{\Delta n}{n}\frac{yz}{r^2},\\ \varpi_1&=\ p\frac{\Delta n}{n}\frac{xy}{r^2},\\ \varpi_2&=\ p\frac{\Delta n}{n}\frac{z^2-x^2}{r^2};\end{aligned}\right\}\quad\ldots\ldots\quad(7)$$

which show that there are in all six directions from O along which there is no scattered ray—two perpendicular to the plane (zx) of original vibration, and four in that plane inclined at angles of 45° to the original ray and its prolongation. No vanishing of the dispersed light in these oblique directions is known from experiment; but before unreservedly discarding the theory which indicates it, we ought to inquire how far our approximation is sufficient to warrant such a step. In neglecting the products of ξ, η, ζ with Δn, we have in reality omitted terms from the result which involve the square and higher powers of Δn, and it may be that the light corresponding to them would not vanish in the specified directions. I have not been able to satisfy myself whether this would be so or not; but I think that, in spite of ignorance on this point, the inference may be safely drawn that the theory is untenable; for the terms in question, depending on the square of the difference of rigidity, are proportional to $\frac{\Delta\mu^2}{\mu^2}$ (where μ is the refractive index), and become of less and less importance as the media approach one another in refrangibility. In the case of particles of mastic suspended in water, the indices are 1·5 and 1·33, and terms depending on the square of Δn must be comparatively small. Yet I could find no indication of a falling off of intensity in the predicted directions in some experiments that I made with precipitated mastic and soap, and accordingly conclude that the hypothesis of a constant density and variable rigidity must be rejected. The only alternative is to suppose, as in the February Number, that the æther preserves its statical properties unchanged when associated with matter, whose effect is therefore merely to increase the inertia of the vibrating parts in greater or less degree.

It may be worth notice that, according to the theory here combated, there would be two polarizing-angles, of $22\frac{1}{2}°$ and $67\frac{1}{2}°$ respectively, when light vibrating in the plane of incidence is reflected from the boundary of two media which differ but little in refrangibility, as may be seen from the reasoning of this paper by remembering that the square of Δn may be neglected. I need scarcely say that in such a case the polarizing angle is really 45°, and that the reflected light does not tend to vanish at the two first-mentioned incidences, whichever way the light may be polarized*.

In equations (5), putting $\Delta n = 0$, we have

* There is a sense in which 45° is the first approximation to the polarizing-angle for all substances. The difference between the true value and 45° may be looked upon as a *correction* depending on the square and higher powers of the difference of optical density.

$$\left.\begin{aligned} \varpi_3 &= 0, \\ \varpi_1 &= p\frac{D'-D}{D}\frac{y}{r}, \\ \varpi_2 &= -p\frac{D'-D}{D}\frac{x}{r}, \end{aligned}\right\} \quad \ldots\ldots (8)$$

which correspond to the results already obtained. Since the optical density is proportional to the square of the refractive index,

$$\frac{D'-D}{D} = \frac{\mu'^2-\mu^2}{\mu^2}. \quad \ldots\ldots (9)$$

In a note to my previous paper I mentioned that no change is required in (8), even though the terms containing the square and higher powers of $\frac{D'-D}{D}$ are retained. As I there showed, the density of a medium may always be supposed to be changed, even in the most arbitrary manner, if suitable bodily forces proportional to the variation of density and to the actual acceleration are conceived to act upon it, while the motion remains absolutely the same as before. The waves thrown off from a small particle which lies in the path of a beam of light are those due to a set of forces proportional to $D'-D$, and parallel to the actual vibrations acting through the space T occupied by the particle. In calculating the effect of the forces, the variation of density is to be taken into account, unless we are content to neglect the square of $D'-D$. But by a second application of our principle we see that the density within the space T may be supposed to be D instead of D', provided we introduce a second set of forces proportional to $D'-D$ and to the acceleration at T. Now it may be proved (Thomson and Tait, p. 569) that the effect of a bodily force applied through a small space T to an elastic medium diminishes without limit with T *even within the region of application*. Accordingly the acceleration at T caused by our first set of forces is of a higher order of magnitude than the forces themselves, and thus, whether $D'-D$ be small or not, the effect of the second set is to be neglected. The error caused by taking in the calculation of the first set the undisturbed instead of the actual acceleration is evidently smaller still.

If it were desired to continue the approximation, some further supposition would be necessary as to the *shape* of the disturbing particles. The leading term, we have seen, depends only on the volume; but the same would not be true for those that follow. However, little exception could be taken to the assumption of a spherical form; and in that case there is no difficulty in proceed-

ing further; but I have not arrived at any results of interest. Without calculation, we may anticipate that, as the diameter of the particles approaches in magnitude the quarter wave-length, the amplitude of the diffracted vibration will begin to increase less rapidly than T, and that about the time the half wave-length is passed an absolute diminution will set in. Of course, when the incident light is compound, the more refrangible elements will be the first to show a sensible deviation from the more simple law.

In his interesting experiments with precipitated vapours, Professor Tyndall* found that when the particles of the cloud illuminated by unpolarized light from the electric lamp had attained such a size that the light discharged normally had lost most of its power of affecting the naked eye with the sensation of colour, even then by analyzing the light with a Nicol placed in its position of minimum transmission the azure could be revived in increased splendour. Professor Tyndall calls this the "residual blue." Experimentally it is doubtless more convenient to analyze the light after diffraction from the cloud; but in theoretical explanation and deduction it is simpler, and comes to the same thing in the end, to consider the original beam as polarized before it falls on the cloud. The residual blue is then the light discharged from the cloud in a direction parallel to that in which the incident light swings. The complete explanation of this and other allied phenomena is yet to be made out; but one thing we learn from our theory, if indeed it is at all to be depended on. However large the particles may be, the light scattered or reflected† parallel to the primary vibrations depends on the square and higher powers of $\frac{D'-D}{D}$, or, in experimental language, of $\frac{\mu'^2-\mu^2}{\mu^2}$. It is easy to see, too, that the first term in the expression of the amplitude must contain a much higher inverse power of λ than λ^{-2}, and that *if it stood alone* it would correspond to a compound light of a much richer colour than that due to very small particles acting in the ordinary way. Still I cannot honestly say that the residual blue is predicted by theory: before the light discharged in this unfavourable direction could become at all sensible, the particles must have grown to such a size that their diameter would bear no inconsiderable proportion to the waves of light; and then we have no right to suppose that the first term in the expansion proceeding by powers of the diameter may be taken

* Phil. Mag. vol. xxxviii. p. 156. Phil. Trans. 1870.

† This may be verified with Fresnel's expression for the intensity of the light regularly reflected when the plane of polarization and plane of incidence include a right angle.

as representing with sufficient approximation the entire series. Indeed the residual blue appears to be rather capricious in its appearance, and to depend on conditions not yet fully known. I may mention that I have not been able to detect any unusually intense coloration in that part of the light from the sky which vibrates in a plane passing through the sun. This is the more remarkable, because it might be supposed that a part at least is light which has twice undergone diffraction, in which case the intensity would vary as λ^{-8} if otherwise undisturbed. But we must not forget that, of the indirect light illuminating the higher strata of our atmosphere, a very considerable fraction must come from the earth itself; and this certainly is coloured any thing but blue. It would be interesting to observe whether the residual light from parts of the sky 90° distant from the sun is in any way dependent on the character of the earth's surface—whether, for example, it is the same as usual over water or when the ground is covered with snow. I presume that with the precipitated clouds there is no question of light diffracted more than once.

Theory would lead us to anticipate that the optical density of the particles of foreign matter may have a large influence on the development of the residual blue. If the particles and the medium in which they are suspended have nearly the same refrangibility, the light emitted parallel to the original vibrations may be expected to be very feeble, not only absolutely, but in comparison with that emitted in other directions. Professor Tyndall's method of precipitating organic vapours (some of which may have a high optical density) in air is then more favourable than the suspension of mastic or other moderately dense solids in water, as used by Brücke and other physicists.

I take this opportunity of referring to the observations of Roscoe on the photographic power of skylight, with which I have only lately become acquainted. The comparison of photographic with luminous intensities is well adapted to exhibit differences of quality related in a simple manner to the wave-length. The very small chemical action of the direct solar rays, as compared with what might have been expected from their intense action on the retina, is a striking verification of the theoretical results developed in the February Number of this Magazine.

LIX. *On the Capillary Phenomena of the common Surface of two Liquids.* *By* G. Quincke.

[Continued from p. 390.]

V. *On the spreading of liquids on other liquids.*

24. If three common surfaces of three liquids intersect in a (curved) line, then three forces operate on a particle of the substance, P, in the line of intersection, which lie in the per-

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X. *On the Electromagnetic Theory of Light.* *By* LORD RAYLEIGH, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

THE claims of the theory propounded by Maxwell, according to which light consists of a disturbance in a medium susceptible of dielectric polarization, are so considerable that it is desirable to extend its application as far as possible to various optical phenomena. The question of the velocity of propagation in vacuum and in singly or doubly refracting transparent dielectrics was considered by Maxwell himself; and the agreement with experiment, though far from perfect, is sufficiently encouraging. More recently it has been shown by Helmholtz †, Lorentz ‡, Fitzgerald §, and J. J. Thomson ‖, that the same theory leads to Fresnel's expressions for the intensity of light reflected and refracted at the surface of separation of transparent media, and that the auxiliary hypotheses necessary in this part of the subject agree with those required to explain the laws of double refraction. In this respect the electromagnetic theory has a marked advantage over the older view, which assimilated luminous vibrations to the ordinary transverse vibrations of elastic solids. According to the latter, Fresnel's laws of double refraction, fully confirmed by modern

* Communicated by the Author.

† Crelle, Bd. lxxii., 1870.

‡ Schlömilch, xxii., 1877.

§ Phil. Trans. 1880.

‖ Phil. Mag. April 1880.

observation*, require us to suppose that in a doubly-refracting crystal the rigidity of the medium varies with the direction of the strain; while, in order to explain the facts relating to the intensities of reflected light, we have to make the inconsistent assumption that the rigidity does not vary in passing from one medium to another. A further discussion of this subject will be found in papers published in the Philosophical Magazine during the year 1871.

If the dielectric medium be endowed with sensible conductivity, the electric vibrations will be damped; that is to say, the light will undergo absorption, with a rapidity which Maxwell has calculated. By supposing the conductivity to be so great that practically complete absorption takes place within a distance comparable with the wave-length, we may obtain a theory of metallic reflection which is not without interest, although the phenomena of abnormal dispersion show that it cannot be regarded as complete.

For an isotropic medium at rest we have the equations (Maxwell's 'Electricity and Magnetism,' §§ 591, 598, 607, 610, 611)

$$u=p+\frac{df}{dt}, \text{ \&c.,} \quad \ldots\ldots \quad (1)$$

$$f=\frac{\mathrm{K}}{4\pi}\mathrm{P}, \text{ \&c.,} \quad \ldots\ldots \quad (2)$$

$$p=\mathrm{CP}, \text{ \&c.,} \quad \ldots\ldots \quad (3)$$

$$\mathrm{P}=-\frac{d\mathrm{F}}{dt}-\frac{d\Psi}{dx}, \text{ \&c.,} \quad \ldots\ldots \quad (4)$$

$$a=\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}, \text{ \&c.,} \quad \ldots\ldots \quad (5)$$

$$a=\mu\alpha, \text{ \&c.,} \quad \ldots\ldots \quad (6)$$

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz}, \text{ \&c.;} \quad \ldots\ldots \quad (7)$$

in which f, g, h are the electric displacements, p, q, r the currents of conduction, u, v, w the total currents, P, Q, R the components of electromotive force, K the specific inductive capacity, C the conductivity, α, β, γ the components of magnetic force, a, b, c the components of magnetization, μ the magnetic capacity, F, G, H the components of electrokinetic momentum, and Ψ the electric potential.

* Glazebrook, Phil. Trans. 1879.

From (2), (4), and (5) we get

$$4\pi\left(\frac{d}{dy}\frac{f}{\mathrm{K}}-\frac{d}{dx}\frac{g}{\mathrm{K}}\right)=-\frac{d}{dt}\left(\frac{d\mathrm{F}}{dy}-\frac{d\mathrm{G}}{dx}\right)=\frac{dc}{dt}\ \&c.\ .\quad(8)$$

In the case of K constant, equation (8) expresses that the electric displacement $\int(f\,dx+g\,dy)$ round a small circuit in the plane of xy corresponds to the electromotive force round the circuit, represented by dc/dt.

Again, from (1), (2), (3), (6), (7),

$$4\pi\left(\frac{df}{dt}+\frac{4\pi\mathrm{C}}{\mathrm{K}}f\right)=\frac{d}{dy}\frac{c}{\mu}-\frac{d}{dz}\frac{b}{\mu}\ \&c.\ .\ .\ .\quad(9)$$

From equations (8) and (9) the problem of reflection can be investigated. In order to limit ourselves to plane waves of simple type, we shall suppose that K, μ, and C are independent of z, and that the electric and magnetic functions are independent of z and (as dependent upon the time) proportional to e^{int}. The two principal cases will be considered separately, (1) when the electric displacements are perpendicular to the plane of incidence, (2) when they are executed in that plane.

Case 1. This is defined by the conditions

$$f=0,\quad g=0,\text{ and (accordingly) } c=0.$$

Thus

$$ina=-4\pi\frac{d}{dy}\frac{h}{\mathrm{K}},\quad inb=4\pi\frac{d}{dx}\frac{h}{\mathrm{K}},\quad\ .\ .\quad(10)$$

$$4\pi\left(in+\frac{4\pi\mathrm{C}}{\mathrm{K}}\right)h=\frac{d}{dx}\frac{b}{\mu}-\frac{d}{dy}\frac{a}{\mu}.\ .\ .\ .\quad(11)$$

Eliminating a and b from (10) and (11), we get

$$\frac{d}{dx}\left(\frac{1}{\mu}\frac{d}{dx}\right)\frac{h}{\mathrm{K}}+\frac{d}{dy}\left(\frac{1}{\mu}\frac{d}{dy}\right)\frac{h}{\mathrm{K}}+n^2\mathrm{K}\left(1-in\frac{4\pi\mathrm{C}}{n}\right)\frac{h}{\mathrm{K}}=0.\quad(12)$$

Case 2. Here the special conditions are

$$h=0,\quad a=0,\quad b=0.$$

We have

$$4\pi\left(\frac{d}{dy}\frac{f}{\mathrm{K}}-\frac{d}{dx}\frac{g}{\mathrm{K}}\right)=inc,\ .\ .\ .\ .\ .\ .\ .\quad(13)$$

$$4\pi\left(in+\frac{4\pi\mathrm{C}}{\mathrm{K}}\right)f=\frac{d}{dy}\left(\frac{c}{\mu}\right),\quad 4\pi\left(in+\frac{4\pi\mathrm{C}}{\mathrm{K}}\right)g=-\frac{d}{dx}\left(\frac{c}{\mu}\right);\quad(14)$$

whence by elimination of f and g,

$$\frac{d}{dx}\left\{\frac{1}{\mathrm{K}n^2(1-4\pi n^{-1}\mathrm{C}\mathrm{K}^{-1})}\frac{d}{dx}\left(\frac{c}{\mu}\right)\right\}$$
$$+\frac{d}{dy}\left\{\frac{1}{\mathrm{K}n^2(1-4\pi n^{-1}\mathrm{C}\mathrm{K}^{-1})}\frac{d}{dy}\left(\frac{c}{\mu}\right)\right\}+\mu\left(\frac{c}{\mu}\right)=0.\quad(15)$$

Equations (12) and (15) simplify considerably in their application to a uniform medium, assuming the common form

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+n^2\mu\mathrm{K}(1-4\pi n\mathrm{C}\mathrm{K}^{-1})=0. \quad . \quad (16)$$

To express the boundary conditions let us suppose that $x=0$ is the surface of transition between two uniform media. From (12) we learn that the required conditions for case 1 are that

$$\frac{h}{\mathrm{K}} \text{ and } \frac{1}{\mu}\frac{d}{dx}\left(\frac{h}{\mathrm{K}}\right)$$

must be continuous.

In like manner, for case 2 we see from (15) that

$$\frac{c}{\mu} \text{ and } \frac{1}{\mathrm{K}(1-4\pi n^{-1}\mathrm{C}\mathrm{K}^{-1})}\frac{d}{dx}\left(\frac{c}{\mu}\right)$$

must be continuous.

If the media are transparent, or but moderately opaque, we have to put $\mathrm{C}=0$. The differential equation is of the form

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+n^2\mu\mathrm{K}=0. \quad . \quad . \quad . \quad . \quad (17)$$

In case 1 the boundary conditions are the continuity of the dependent variable and of $\frac{1}{\mu}\frac{d}{dx}$, and in case 2 the continuity of the dependent variable and of $\frac{1}{\mathrm{K}}\frac{d}{dx}$. Analytically, the results are thus of the same form in both cases. If θ and θ_1 are respectively the angles of incidence and refraction, the ratio of the reflected to the incident vibration is in case 1

$$\frac{\frac{\tan\theta_1}{\tan\theta}-\frac{\mu}{\mu_1}}{\frac{\tan\theta_1}{\tan\theta}+\frac{\mu}{\mu_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and in case 2

$$\frac{\frac{\tan\theta_1}{\tan\theta}-\frac{\mathrm{K}}{\mathrm{K}_1}}{\frac{\tan\theta_1}{\tan\theta}+\frac{\mathrm{K}}{\mathrm{K}_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which K, μ relate to the first, and K_1, μ_1 to the second medium; while the relation between θ_1 and θ is

$$\mathrm{K}_1\mu_1 : \mathrm{K}\mu = \sin^2\theta : \sin^2\theta_1. \quad . \quad . \quad . \quad . \quad (20)$$

As Helmholtz has remarked, Fresnel's formulæ may be

obtained on two distinct suppositions. If $\mu_1=\mu$,

$$(18) = \frac{\sin(\theta_1-\theta)}{\sin(\theta_1+\theta)},$$

and

$$(19) = \frac{\tan(\theta_1-\theta)}{\tan(\theta_1+\theta)};$$

but if $K_1=K$, then (19) identifies itself with the sine-formula, and (18) with the tangent-formula. Electrical phenomena, however, lead us to prefer the former alternative, and thus to the assumption that the *electric* displacements are perpendicular to the plane of polarization. The formulæ for the refracted waves, which follow from those of the reflected waves in virtue of the principle of energy alone, do not call for detailed consideration.

In the problem of perpendicular incidence, we have from (12), if μ be constant and C zero,

$$\frac{d^2}{dx^2}\frac{h}{K}+n^2\mu K\left(\frac{h}{K}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For an application of this equation to determine the influence of defective suddenness in the transition between two uniform media, the reader is referred to a paper in the eleventh volume of the Proceedings of the Mathematical Society.

In order to obtain a theory of metallic reflection, C must be considered to have a finite value in the second medium. The symbolical solution is not thereby altered from that applicable to transparent media, the effect of the finiteness of C being completely represented in both cases by the substitution of $K(1-i4\pi nCK^{-1})$ for K. Thus, if μ be constant, the formula for the amplitude and phase of the reflected wave in case 1 is to be found by transformation of (18), in which the imaginary angle of refraction θ_1 is connected with θ by the relation

$$K_1(1-i4\pi nCK_1^{-1}) : K = \sin^2\theta : \sin^2\theta_1. \quad . \quad . \quad (22)$$

In like manner the solution for case 2 is to be found by transformation of (19) under the same supposition.

With regard to the proposed transformations, the reader is referred to a paper by Eisenlohr* and to some remarks thereupon by myself†. The results are the formulæ published without proof by Cauchy. From the calculations of Eisenlohr it appears that Jamin's observations cannot be reconciled with the formulæ without supposing $K_1 : K$, *i. e.* the real part of the square of the complex refractive index, to be negative—a

* Pogg. *Ann.* t. civ. p. 368. † Phil. Mag. May 1872.

further proof that much remains to be done before the electrical theory of metallic reflection can be accepted as complete*.

The same fundamental equations (8) and (9) will now be applied to the problem of determining the effect on a train of plane waves of a small variation in the quantities K and μ which define the medium. A similar method will be adopted to that already used for light in a paper "On the Scattering of Light by small Particles"†, and in my book 'On the Theory of Sound,' § 296, the principle of which consists in an approximation depending upon the neglect of the higher powers of the small variations ΔK and $\Delta\mu$.

Let us suppose that a train of plane waves, in which the electric displacement is parallel to z, and magnetization parallel to y, propagates itself parallel to x undisturbed until it falls upon a region where the generally constant values of K and μ become $K+\Delta K$ and $\mu+\Delta\mu$. If ΔK and $\Delta\mu$ were zero, the wave would pass on as before; but under the circumstances secondary waves are generated, which diverge from the region of disturbance, and are ultimately, when ΔK and $\Delta\mu$ are small enough, proportional in magnitude to these quantities. As the expression of the primary waves we may take

$$h_0 = e^{int}e^{ikx}, \quad \ldots\ldots\ldots \quad (23)$$

and corresponding thereto, by (8),

$$b_0 = 4\pi k n^{-1} K^{-1} e^{int}e^{ikx}, \quad \ldots\ldots \quad (24)$$

in which, if λ denote the wave-length, $k=2\pi/\lambda$, and n/λ is the velocity of propagation $(K\mu)^{-\frac{1}{2}}$. The complete values of the functions being represented, as before, by f, g, h, a, b, c, we shall put

$$f=f_0+f_1+f_2+\ldots\&c., \quad a=a_0+a_1+\ldots\&c.,$$

$f_0\ldots a_0\ldots$ being independent of ΔK and $\Delta\mu$, $f_1\ldots a_1\ldots$ being of the first order, $f_2\ldots a_2\ldots$ of the second order, and so on, in these quantities. In the actual case f_0, g_0, a_0, c_0 vanish, and only h_0 and b_0 are finite.

From (8) and (9) with $C=0$, we get

* July 15.—I see that Lorentz, in a pamphlet *Over de Theorie der Terugkaatsing en Breking van het Licht* (Arnhem, 1875), has developed a theory of metallic reflection similar to that indicated in the text, and has noticed the same difficulty in the application to experiment.

† Phil. Mag. June 1871.

$$\left.\begin{aligned}
4\pi\left\{\frac{df}{dy}-\frac{dg}{dx}+\mathrm{K}\frac{d}{dy}(f\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dx}(g\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{dc}{dt},\\
4\pi\left\{\frac{dg}{dz}-\frac{dh}{dy}+\mathrm{K}\frac{d}{dz}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dy}(h\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{da}{dt},\\
4\pi\left\{\frac{dh}{dx}-\frac{df}{dz}+\mathrm{K}\frac{d}{dx}(h\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dz}(f\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{db}{dt}.
\end{aligned}\right\}\quad(25)$$

$$\left.\begin{aligned}
\frac{dc}{dy}-\frac{db}{dz}+\mu\frac{d}{dy}(c\Delta\mu^{-1})-\mu\frac{d}{dz}(b\Delta\mu^{-1})&=4\pi\mu\frac{df}{dt},\\
\frac{da}{dz}-\frac{dc}{dx}+\mu\frac{d}{dz}(a\Delta\mu^{-1})-\mu\frac{d}{dx}(c\Delta\mu^{-1})&=4\pi\mu\frac{dg}{dt},\\
\frac{db}{dx}-\frac{da}{dy}+\mu\frac{d}{dx}(b\Delta\mu^{-1})-\mu\frac{d}{dy}(a\Delta\mu^{-1})&=4\pi\mu\frac{dh}{dt}.
\end{aligned}\right\}\quad(26)$$

By differentiation of the first equation of (26) and substitution from (25), we get, having regard to

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,\quad(27)$$

which is a consequence of (1), (3), (7),

$$\begin{aligned}
\mu\mathrm{K}\frac{d^2f}{dt^2}=\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}+\mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f\Delta\mathrm{K}^{-1})\\
-\mathrm{K}\frac{d^2}{dx\,dy}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d^2}{dx\,dz}(h\Delta\mathrm{K}^{-1})\\
+\frac{\mu\mathrm{K}}{4\pi}\frac{d^2}{dy\,dt}(\mathrm{C}\Delta\mu^{-1})-\frac{\mu\mathrm{K}}{4\pi}\frac{d^2}{dz\,dt}(b\Delta\mu^{-1}),
\end{aligned}$$

or, remembering that the functions as dependent upon time vary as e^{int},

$$\begin{aligned}
\nabla^2f+k^2f+\mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f\Delta\mathrm{K}^{-1})\\
-\mathrm{K}\frac{d^2}{dx\,dy}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d^2}{dx\,dz}(h\Delta\mathrm{K}^{-1})\\
+\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dy}(c\Delta\mu^{-1})-\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dz}(b\Delta\mu^{-1})=0,
\end{aligned}\quad(28)$$

with two similar equations in g and h.

Introducing now the expansion in powers of $\Delta\mathrm{K}$ and $\Delta\mu$, we get as the first approximation

$$\nabla^2f_1+k^2f_1-\mathrm{K}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1})-\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dz}(b_0\Delta\mu^{-1})=0,$$

or, on substitution for b_0 in terms of h_0 from (23), (24),

$$\nabla^2 f_1 + k^2 f_1 - \mathrm{K}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1}) - ik\mu\frac{d}{dz}(h_0\Delta\mu^{-1}) = 0, \quad (29)$$

and

$$\nabla^2 g_1 + k^2 g_1 - \mathrm{K}\frac{d^2}{dy\,dz}(h_0\Delta\mathrm{K}^{-1}) = 0, \quad \ldots\ldots \quad (30)$$

$$\nabla^2 h_1 + k^2 h_1 + \mathrm{K}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)(h_0\Delta\mathrm{K}^{-1}) + ik\mu\frac{d}{dx}(h_0\Delta\mu^{-1}) = 0. \quad \ldots \quad (31)$$

The solution of (29) is

$$f_1 = -\frac{\mathrm{K}}{4\pi}\iiint\frac{e^{-ikr}}{r}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1})\,dx\,dy\,dz - \frac{ik\mu}{4\pi}\iiint\frac{e^{-ikr}}{r}\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\,dx\,dy\,dz, \quad \ldots \quad (32)$$

where r, equal to $\sqrt{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}}$, is the distance of the element of volume $dx\,dy\,dz$ from the point α, β, γ at which f_1 is to be estimated.

In applying (32) to the calculation of a secondary wave at a distance from the region of disturbance, we may conveniently integrate it by parts. Thus,

$$f_1 = -\frac{\mathrm{K}}{4\pi}\iiint h_0\Delta\mathrm{K}^{-1}\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz + \frac{ik\mu}{4\pi}\iiint h_0\Delta\mathrm{K}^{-1}\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz.$$

From the general value of r,

$$\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\gamma-z}{r}\frac{e^{-ikr}(1+ikr)}{r^2}, \quad \ldots\ldots \quad (33)$$

$$\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\alpha-x}{r}\frac{\gamma-z}{r}\frac{e^{-ikr}(3+3ikr-k^2r^2)}{r^3}. \quad \ldots \quad (34)$$

If r be sufficiently great in comparison with λ, only the highest power of kr in the above expressions need be retained; and if r be also great in comparison with the dimensions of the region of disturbance, supposed to be situated about the origin of coordinates, $(\alpha-x)/r$ &c. may be replaced by α/r &c. Thus,

$$\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\gamma}{r}\frac{ike^{-ikr}}{r};$$

$$\frac{d^2}{dz\,dx}\left(\frac{e^{-ikr}}{r}\right) = -\frac{\alpha\gamma}{r^2}\frac{k^2e^{-ikr}}{r};$$

and the expression for f_1 becomes

$$f = \frac{k^2}{4\pi r}\Big[\mathrm{K}\frac{\alpha\gamma}{r^2}\iiint h_0 \Delta \mathrm{K}^{-1} e^{-ikr}\, dx\, dy\, dz - \mu\frac{\gamma}{r}\iiint h_0\, \Delta\mu^{-1} e^{-ikr}\, dx\, dy\, dz\Big].$$

For the sake of brevity we will write this

$$f_1 = \frac{k^2}{4\pi r}\Big[\mathrm{KP}\frac{\alpha\gamma}{r^2} - \mu\mathrm{Q}\frac{\gamma}{r}\Big], \quad . \quad . \quad . \quad (35)$$

where

$$\left.\begin{aligned} \mathrm{P} &= \iiint h_0 \Delta \mathrm{K}^{-1} e^{-ikr}\, dx\, dy\, dz, \\ \mathrm{Q} &= \iiint h_0 \Delta\mu^{-1} e^{-ikr}\, dx\, dy\, dz. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36)$$

In like manner from (30) and (31),

$$g_1 = \frac{k^2}{4\pi r}\Big[\mathrm{KP}\frac{\beta\gamma}{r^2}\Big], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$h_1 = \frac{k^2}{4\pi r}\Big[-\mathrm{KP}\frac{\alpha^2+\beta^2}{r^2} + \mu\mathrm{Q}\frac{\alpha}{r}\Big]. \quad . \quad . \quad . \quad (38)$$

Equations (35), (37), (38) express the electric displacement in the secondary waves. Since $\alpha f + \beta g + \gamma h = 0$, the displacement is perpendicular to the direction of the secondary ray. The general expression for the intensity is found by adding the squares of f, g, h; but it will be sufficient for our present purpose to limit ourselves to the case where the secondary ray is perpendicular to the primary ray, *i. e.* to the case $\alpha = 0$. Then

$$f^2 + g^2 + h^2 = \frac{k^4}{16\pi^2 r^2}\Big[\mathrm{K}^2\mathrm{P}^2\frac{\beta^2}{r^2} + \mu^2\mathrm{Q}^2\frac{\gamma^2}{r^2}\Big]. \quad . \quad . \quad (39)$$

If P and Q are both finite, there is no direction along which the secondary light vanishes. We find by experiment, however, that the light scattered by small particles on which polarized light impinges does vanish in one direction perpendicular to the original ray; and thus either P or Q must vanish. Now, when the particles are very small, we have

$$\mathrm{P} = h_0 \Delta \mathrm{K}^{-1} e^{-ikr}\iiint dx\, dy\, dz, \quad \mathrm{Q} = h_0 \Delta\mu^{-1} e^{-ikr}\iiint dx\, dy\, dz; \quad (40)$$

so that if P vanishes, $\Delta \mathrm{K} = 0$; and if Q vanishes, $\Delta\mu = 0$. The optical evidence that either $\Delta \mathrm{K}$ or $\Delta\mu$ vanishes is thus very strong; while electrical reasons lead us to conclude that it is $\Delta\mu$.

If we write T for the volume of the small particle, we get

from (40), as the special forms of (35), (37), (38) applicable to this case,

$$f_1 = \frac{\pi T}{\lambda^2 r} e^{i(nt-kr)}\left[K\Delta K^{-1}\frac{\alpha\gamma}{r^2} - \mu\Delta\mu^{-1}\frac{\gamma}{r}\right], \quad . \quad . \quad (41)$$

$$g_1 = \frac{\pi T}{\lambda^2 r} e^{i(nt-kr)}\left[K\Delta K^{-1}\frac{\beta\gamma}{r^2}\right], \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$h_1 = \frac{\pi T}{\lambda^2 r} e^{i(nt-kr)}\left[-K\Delta K^{-1}\frac{\alpha^2+\beta^2}{r^2} + \mu\Delta\mu^{-1}\frac{\alpha}{r}\right]. \quad (43)$$

If $\Delta\mu = 0$, as we shall henceforward suppose, $f : g = \alpha : \beta$, showing that the electrical displacement is in the plane containing the secondary ray and the direction of primary electrical displacement (z), and

$$f_1^2 + g_1^2 + h_1^2 \propto \frac{\alpha^2+\beta^2}{r^2};$$

so that the intensity is proportional to the square of the sine of the angle between the secondary ray and the direction of the primary electrical displacement. The blue colour of the light scattered from small particles is explained by the occurrence of λ^2 in the denominators of the expressions for f_1, g_1, h_1; but for further particulars on this subject the reader must be referred to my previous papers.

Equations (35), (36), &c. are rigorously applicable, however large the region of disturbance, if the square of ΔK may really be neglected. From them we see that, under the circumstances in question, each element of a homogeneous obstacle acts independently as a centre of disturbance, and that the aggregate effect in any direction depends upon the phases of the elementary secondary disturbances as affected by the situation of the element along the paths of the primary and of the secondary light. In fact,

$$P = \Delta K^{-1} e^{int} \iiint e^{ikx} e^{-ikr}\, dx\, dy\, dz.$$

If θ, ϕ be the angles defining (in the usual notation) the direction of the secondary ray, and r_0 correspond to the origin of coordinates, we have

$$P = \Delta K^{-1} e^{i(nt-kr_0)} \iiint e^{ik(x+x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)}\, dx\, dy\, dz; \quad (44)$$

and the question now before us for consideration is the value of the integral in (44) as dependent upon the size of the obstacle and the direction of the secondary ray. It is evident that the formulæ are applicable only when the whole retardation of the primary light in traversing the obstacle can be neglected in comparison with the wave-length; but if this

condition be satisfied, there is no further limitation upon the size of the obstacle. In the case where the secondary ray forms the prolongation of the primary, or deviates sufficiently little from this direction, the exponential in (44) reduces to unity, signifying that every element of the obstacle acts alike, any retardation of phase at starting due to situation along the primary ray being balanced by an acceleration corresponding to a less distance to be travelled along the secondary ray. At a greater or less obliquity, according to the size of the obstacle, opposition of phase sets in; and at still greater obliquities the resultant can be found only by an exact integration. Its intensity is then less, and generally much less, than in the first case—a conclusion abundantly borne out by observation.

The simplest example of this kind is that afforded by an infinite cylinder (*e. g.* a fine spider-line), on which the light impinges perpendicularly to the axis, so that every thing takes places in two dimensions. This case is indeed not strictly covered by the preceding formulæ, on account of the infinite extension of the region of disturbance; but a moment's consideration will make it clear that each elementary column here acts according to the laws already described—that is to say, gives rise to a component disturbance whose phase is determined by the situation of the element along the primary and secondary rays. If the angle between the two rays be called χ, we have to consider the value of

$$\iint e^{ik(x+x\cos\chi+y\sin\chi)}\,dx\,dy.$$

Introducing polar coordinates r, θ, we find

$$x+x\cos\chi+y\sin\chi=2r\cos\tfrac{1}{2}\chi\ \cos(\theta-\tfrac{1}{2}\chi);$$

so that the integral

$$\begin{aligned}
&=\iint e^{ikr\,.\,2\cos\frac{1}{2}\chi\,.\,\cos\theta}\,r\,dr\,d\theta\\
&=\int_0^a\int_0^{2\pi}\{\cos(2kr\cos\tfrac{1}{2}\chi\ \cos\theta)\\
&\qquad\qquad +i\sin(2kr\cos\tfrac{1}{2}\chi\ \cos\theta)\}\,r\,dr\,d\theta\\
&=2\pi\int_0^a \mathrm{J}_0(2kr\cos\tfrac{1}{2}\chi)\,r\,dr, \quad\ldots\ldots\quad (45)
\end{aligned}$$

J_0 denoting the Bessel's function of zero order.

The integration with respect to r indicated in (45) can be effected by known properties of Bessel's functions; and the result is expressible by a function of the first order. We get

$$\frac{\pi a}{k\cos\frac{1}{2}\chi}\mathrm{J}_1(2ka\cos\tfrac{1}{2}\chi);\quad\ldots\ldots\quad (46)$$

and J_1 is defined by

$$J_1(z)=\frac{z}{2}\left(1-\frac{z^2}{2.4}+\frac{z^4}{2.4^2.6}-\frac{z^6}{2.4^2.6^2.8}+\dots\right). \quad (47)$$

If $\cos\frac{1}{2}\chi=0$ (*i. e.* in the direction of original propagation), (46) becomes πa^2, every element of the area acting alike. This is the maximum value. When χ is such that

$$2ka\cos\tfrac{1}{2}\chi=\pi\times 1{\cdot}2197,$$

the secondary light vanishes, at a greater angle revives, then vanishes again, and so on, the angles being of course functions of the wave-length. If we conceive the cylinder to increase in size gradually from zero, the scattered light vanishes first in the backward direction $\chi=0$, in which direction evidently the greatest differences of phase occur. Every thing is determined by the course of the function J_1; and (46) within the limits of its application embodies the theory of Young's eriometer.

We will now consider the case of an obstacle in the form of a sphere. If z be a coordinate measured perpendicularly to the plane containing the primary and secondary rays, formula (46), multiplied by dz, will represent the effect of a slice of the sphere, whose radius is a and thickness dz, and what remains to be effected is merely the integration with respect to z. For this purpose we write $z=c\sin\phi$, $a=c\cos\phi$, where c is the radius of the sphere. The integral then takes the form

$$\frac{2\pi c^2}{k\cos\frac{1}{2}\chi}\int_0^{\frac{1}{2}\pi}J_1(2kc\cos\tfrac{1}{2}\chi\cos\phi)\cos^2\phi\,d\phi, \quad . \quad (48)$$

or, if we expand J_1 by (47), and integrate according to a known formula,

$$\frac{2\pi c^3}{3}\left\{2-\frac{m^2}{5}+\frac{m^4}{7.5.4}-\frac{m^6}{9.7.5.4.6}+\frac{m^8}{11.9.7.5.4.6.8}\cdots\right\}, \quad (49)^*$$

in which m is written for $2kc\cos\frac{1}{2}\chi$. It will be understood that (49), after multiplication by $e^{int}\Delta K^{-1}$, gives merely the value of P in (36), and that to find the complete expression for the secondary light in any direction other factors must be introduced in accordance with (35), (37), (38). The angle χ,

* July 15.—I find for the first root of (49), $m=4{\cdot}50$, giving as the smallest obliquity $(\pi-\chi)$ at which the secondary light vanishes,

$$\pi-\chi=2\sin^{-1}(4{\cdot}50/2kc).$$

being that included between the secondary ray and the axis of x, may be expressed by

$$\sin\chi = \sqrt{(\beta^2+\gamma^2)} \div r. \quad . \quad . \quad . \quad . \quad (50)$$

Our theory, as hitherto developed, shows that, whatever the shape and size of the particles, there is no scattered light in a direction parallel to the primary electric displacements, except such as may depend upon squares and higher powers of the difference of optical properties. In order to render an account of the "residual blue" observed by Tyndall when particles in their growth have reached a certain magnitude, it is necessary to pursue the approximation. By (28), with $\Delta\mu$ neglected, we have

$$\nabla^2 f_2 + k^2 f_2 + \mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f_1\Delta\mathrm{K}^{-1}) - \mathrm{K}\frac{d^2}{dx\,dy}(g_1\Delta\mathrm{K}^{-1}) - \mathrm{K}\frac{d^2}{dx\,dz}(h_1\Delta\mathrm{K}^{-1}) = 0, \quad (51)$$

and two similar equations in g_2 and h_2. On the supposition that f_1, g_1, h_1 are known throughout the region of disturbance, these equations may be solved in the same way as (29), (30), and (31). For the sake of brevity we may confine ourselves to the particular direction for which the terms of the first order vanish. Thus at a sufficient distance r' along the axis of z,

$$f_2 = -\frac{k^2\mathrm{K}}{4\pi r'}\iiint f_1\Delta\mathrm{K}^{-1}e^{-ikr'}\,d\alpha\,d\beta\,d\gamma, \quad . \quad . \quad (52)$$

$$g_2 = -\frac{k^2\mathrm{K}}{4\pi r'}\iiint g_1\Delta\mathrm{K}^{-1}e^{-ikr'}\,d\alpha\,d\beta\,d\gamma, \quad . \quad . \quad (53)$$

$$h_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

We have now to find the values of f_1 and g_1 within the region of disturbance, to which of course (35) &c. are not applicable. In the general solution (32), h_0 is a function of x only; so that the elements of the integral vanish in the *interior* of a homogeneous obstacle, and we have only to deal with the surface. Integrating by parts across this surface, we find

$$f_1 = \frac{\mathrm{K}}{4\pi}\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz = -\frac{\mathrm{K}}{4\pi}\frac{d}{d\alpha}\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\cdot\frac{e^{-ikr}}{r}\,dx\,dy\,dz, \quad (55)$$

r being a function of x and α only through $(\alpha - x)$. In like

manner

$$g_1 = -\frac{\mathrm{K}}{4\pi}\frac{d}{d\beta}\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\cdot\frac{e^{-ikr}}{r}\,dx\,dy\,dz. \quad . \quad . \quad (56)$$

In the case of a small homogeneous sphere, whose centre is taken as origin of coordinates, these formulæ lead to fairly simple results. The triple integral in (55), (56) may readily be exhibited in its real character of a surface-integral. Thus

$$\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{e^{-ikr}}{r}\,dx\,dy\,dz = -\Delta\mathrm{K}^{-1}\iint\frac{h_0 z}{c}\frac{e^{-ikr}}{r}\,d\mathrm{S}, \quad (57)$$

where $d\mathrm{S}$ is an element of the surface whose radius is c. This applies to a sphere of any size; but we have now to introduce an approximation depending on the supposition that kc is small. As far as the first power of kc,

$$-\Delta\mathrm{K}^{-1}\iint\frac{h_0 z}{c}\frac{e^{-ikr}}{r}\,d\mathrm{S} = -\Delta\mathrm{K}^{-1}\frac{e^{int}}{c}\iint\left(\frac{z+ikzx}{r}-ikz\right)d\mathrm{S}$$

$$= -\Delta\mathrm{K}^{-1}\frac{e^{int}}{c}\iint\frac{z+ikzx}{r}\,d\mathrm{S},$$

in which the double integral is the common potential of matter distributed over the spherical surface with density $(z+ikzx)$. Calling this for the moment V, we have (Thomson and Tait, 'Nat. Phil.' § 536) at any internal point (α, β, γ),

$$\mathrm{V} = 4\pi c(\gamma + \tfrac{1}{3}ik\gamma\alpha);$$

so that

$$\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{e^{-ikr'}}{r}\,dx\,dy\,dz$$

$$= -4\pi\Delta\mathrm{K}^{-1}e^{int}(\gamma + \tfrac{1}{3}ik\gamma\alpha). \quad . \quad (58)$$

Thus by (55), (56),

$$f_1 = \tfrac{1}{3}\mathrm{K}\Delta\mathrm{K}^{-1}ik\gamma e^{int}, \quad g_1 = 0. \quad . \quad . \quad . \quad . \quad (59)$$

We are now prepared to calculate f_2, g_2 from (52), (53). These formulæ apply to both directions along the axis of z; but in what follows it will be convenient to suppose that it is the positive direction which is under consideration. In this case, if ρ denote the distance from the centre of the sphere, $r' = \rho - \gamma$ and $e^{-ikr'} = e^{-ik\rho}(1+ik\gamma)$ approximately; so that

$$f_2 = -\frac{k^2(\mathrm{K}\Delta\mathrm{K}^{-1})^2 e^{i(nt-k\rho)}}{12\pi\rho}\iiint ik\gamma(1+ik\gamma)\,d\alpha\,d\beta\,d\gamma$$

$$= \frac{k^4(\mathrm{K}\Delta\mathrm{K}^{-1})^2}{12\pi}\frac{e^{i(nt-k\rho)}}{\rho}\iiint\gamma^2\,d\alpha\,d\beta\,d\gamma;$$

or if, as before, T be the volume of the sphere,

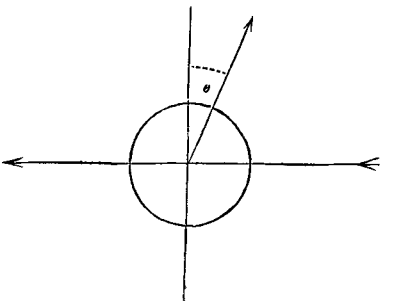
$$\left.\begin{aligned} f_2 &= \frac{\pi \mathrm{T}}{\lambda^2 \rho} e^{i(nt-kp)} (\mathrm{K}\Delta\mathrm{K}^{-1})^2 \frac{k^2c^2}{15}, \\ g_2 &= 0. \end{aligned}\right\} \quad . \quad . \quad (60)$$

Comparing (60) and (41), we see that the amplitude of the light scattered along z is not only of higher order in $\Delta\mathrm{K}$, but is also of the order k^2c^2 in comparison with that scattered in other directions. The incident light being white, the intensity of the component colours scattered along z varies as the inverse 8th power of the wave-length, so that the resultant light is a rich blue.

There is another point of importance to be noticed. Although when the terms of the second order are included the scattered light does not vanish along the axis of z, the peculiarity is not lost, but merely transferred to another direction. Putting together the terms of the first and second orders, we see that the scattered light will vanish in a direction in the plane of xz, inclined to z (towards x) at a small angle θ, such that

$$\theta = -\mathrm{K}\Delta\mathrm{K}^{-1} \frac{k^2c^2}{15} = \frac{\Delta\mathrm{K}}{\mathrm{K}} \frac{k^2c^2}{15}. \quad . \quad . \quad . \quad (61)$$

In the usual case of particles optically denser than the surrounding medium, $\Delta\mathrm{K}$ is positive, from which we gather that the direction in which the scattered light vanishes to the second order of approximation is inclined backwards, so that the angle through which the light may be supposed to be bent by the action of the particle is *obtuse*.



The fact that, when the primary light is polarized, there is in one perpendicular direction no light scattered by very small particles, was stated by Stokes*; but it is, I believe, to Tyndall that we owe the observation that with somewhat larger particles the direction of minimum illumination becomes oblique. I do not find, however, any record of the direction of the obliquity (that is, of the sign of the small angle θ), and have therefore made a few observations for my own satisfaction.

In a darkened room a beam of sunlight was concentrated

* Phil. Trans. 1852, § 183.

by a large lens of 2 or 3 feet focus; and in the path of the light was placed a beaker glass, containing a dilute solution of hyposulphite of soda. On the addition of a small quantity of dilute sulphuric acid a precipitate of sulphur slowly forms, and during its growth manifests exceedingly well the phenomena under consideration. The more dilute the solutions, the slower is the progress of the precipitation. A strength such that there is a delay of four or five minutes before any effect is apparent, will be found suitable; but no great nicety of adjustment is necessary. By addition of ammonia in sufficient quantity to neutralize the acid, the precipitation may be arrested at any desired stage. More time is thus obtained to complete the examination; but the condition of things is not absolutely permanent, the already precipitated sulphur appearing to aggregate itself into large masses.

In the optical examination we may, if we prefer it, polarize the primary light; but it is usually more convenient to analyze the scattered light. In the early stages of the precipitation the polarization is complete in a perpendicular direction, and incomplete in oblique directions. After an interval the polarization begins to be incomplete in the perpendicular direction, the light which reaches the eye when the nicol is in the position of minimum transmission being of a beautiful blue, much richer than any thing that can be be seen in the earlier stages. This is the moment to examine whether there is a more complete polarization in a direction somewhat oblique; and it is found that with θ positive there is in fact an oblique direction of more complete polarization, while with θ negative the polarization is more imperfect than in the perpendicular direction itself.

The polarization in a distinctly oblique direction, however, is not perfect, a feature for which more than one reason may be put forward. In the first place, with a given size of particles, the direction of complete polarization indicated by (61) is a function of the colour of the light, the value of θ being three or four times as large for the violet as for the red end of the spectrum. The experiment is, in fact, much improved by passing the primary light through a coloured glass held in the window-shutter. Not only is the oblique direction of maximum polarization more definite and the polarization itself more complete, but the observation is easier than with white light, by the uniformity of the colour of the light scattered in various directions. If we begin with a blue glass, we may observe the gradually increasing obliquity of the direction of maximum polarization; and then by exchanging the blue

glass for a red one, we may revert to the original condition of things, and observe the transition from perpendicularity to obliquity over again. The change in the wave-length of the light has the same effect as a change in the size of the particles; and the comparison gives curious information as to the rate of growth.

But even with homogeneous light it would be unreasonable to expect an oblique direction of perfect polarization. So long as the particles are all very small in comparison with the wave-length, there is complete polarization in the perpendicular direction; but when the size is such that obliquity sets in, the degree of obliquity will vary with the size of the particles, and the polarization will be complete only on the very unlikely condition that the size is the same for them all. It must not be forgotten, too, that a very moderate increase in dimensions may carry the particles beyond the reach of our approximations.

The fact that at this stage the polarization is a maximum when the angle through which the light is turned *exceeds* a right angle is the more worthy of note, as the opposite result would probably have been expected. By Brewster's law this angle in the case of a plate is *less* than a right angle; so that not only is the law of polarization for a very small particle different from that applicable to a plate, but the first effect of an increase of size is to augment the difference.

We must remember that our recent results are limited to particles of a spherical form. It is not difficult to see that, for elongated particles, the terms in $(\Delta K)^2$ may be of the same order with respect to kc as the principal term; so that if $(\Delta K)^2$ be sensible, mere smallness of the particle will not secure complete evanescence of scattered light along z. The general solution of the problem for an infinitesimal particle of arbitrary shape must raise the same difficulties as beset the general determination of the induced magnetism developed in a piece of soft iron when placed in a uniform field of force. In the case of an ellipsoidal particle the problem is soluble; but it is perhaps premature to enter upon it, until experiment has indicated the existence of phenomena likely to be explained thereby.

For an infinitesimal particle in the form of a sphere, we may readily obtain the complete solution without any approximation depending upon the smallness of ΔK. We know by the analogous theory of magnetism, that a dielectric sphere situated in a uniform field of electric force will undergo electric displacement of uniform amount, and in a direction parallel to that of the force. Thus the complete solution applicable to

an infinitely small sphere is obtained from (29), (30), (31) by writing h for h_0; where by h is denoted the actual displacement (parallel to z) within the particle, and by h_0 the displacement in the enveloping medium under the same electric force. If K′ be the specific inductive capacity for the particle, the ratio of $h : h_0$ is $3K' : K'+2K$; and in this ratio the results expressed in (41), (42), (43) are to be increased. If we extract the factors $K\Delta K^{-1}$ which there occur, we get

$$\frac{3K'}{K'+2K}K\Delta K^{-1}=\frac{3K'K}{K'+2K}\left(\frac{1}{K'}-\frac{1}{K}\right)=-\frac{3(K'-K)}{K'+2K};$$

so that

$$f=-\frac{3(K'-K)}{K'+2K}\frac{\pi T}{\lambda^2 r}\frac{\alpha\gamma}{r^2}e^{i(nt-kr)}, \&c. \quad . \quad . \quad . \quad (62)$$

We learn from (62) that our former result as to the evanescence of the secondary light along z is true for an infinitely small spherical particle to *all* orders of ΔK.

We will now return to the two-dimension problem with the view of determining the disturbance resulting from the impact of plane waves upon a cylindrical obstacle whose axis is parallel to the plane of the waves. There are, as in the problem of reflection from plane surfaces, two principal cases—(1) when the electric displacements are parallel to the axis of the cylinder taken as axis of z, (2) when the electric displacements are perpendicular to this direction.

Case 1.—From (12), with $C=0$, $\mu=$ constant,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\frac{h}{K}+n^2\mu K\frac{h}{K}=0;$$

or if, as before, $k=2\pi/\lambda$,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+k^2\right)\frac{h}{K}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

in which k is constant in each medium, but changes as we pass from one medium to another. From (63) we see that the problem now before us is analytically identical with that treated in my book on Sound, § 343, to which I must refer for more detailed explanations. The incident plane waves are represented by

$$e^{int}e^{ikx}=e^{int}e^{ikr\cos\theta}$$

$$=e^{int}\{J_0(kr)+2iJ_1(kr)\cos\theta+\ldots+2i^mJ_m(kr)\cos m\theta+\ldots\}; \quad (64)$$

and we have to find for each value of m an internal motion finite at the centre, and an external motion representing a divergent wave, which shall in conjunction with (64) satisfy at the surface of the cylinder ($r=c$) the condition that the

function and its differential coefficient with respect to r shall be continuous. The divergent wave is expressed by

$$B_0\psi_0 + B_1\psi_1 \cos\theta + B_2\psi_2 \cos 2\theta + \dots,$$

where ψ_0, ψ_1, &c. are the functions of kr defined in § 341. The coefficients B are determined in accordance with

$$B_m \left\{ kc \frac{d\psi_m}{d.kc} J_m(k'c) - k'c\, \psi_m \frac{d}{d.k'c} J_m(k'c) \right\}$$
$$= 2i^m \{k'c J_m(kc) J_m'(k'c) - kc J_m(k'c) J_m'(kc)\},$$

except in the case of $m=0$, when $2i^m$ on the right-hand side is to be replaced by i^m. In working out the result we suppose kc and $k'c$ to be small; and we find approximately for the secondary disturbance corresponding to (64)

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[\frac{k'^2c^2 - k^2c^2}{2} - \frac{k^2c^2(k'^2c^2 - k^2c^2)}{8} \cos\theta \right]; \quad (65)$$

showing, as was to be expected, that the leading term is independent of θ.

For case 2, which is of greater interest, we have from (15),

$$\left(\frac{d}{dx}\frac{1}{k^2}\frac{d}{dx} + \frac{d}{dy}\frac{1}{k^2}\frac{d}{dy} + 1 \right) c = 0. \quad . \quad . \quad (66)^*$$

This is of the same form as (63) within a uniform medium, but gives a different boundary condition at a surface of transition. In both cases the function itself is to be continuous; but in that with which we are now concerned the second condition requires the continuity of the differential coefficient *after division by* k^2. The equation for B_m is therefore

$$B_m \left\{ k'c \frac{d\psi_m}{d.kc} J_m(k'c) - kc\, \psi_m \frac{dJ_m(k'c)}{d.k'c} \right\}$$
$$= 2i_m \{kc\, J_m(kc)\, J_m'(k'c) - k'c\, J_m(k'c)\, J_m'(kc)\},$$

with the understanding that the 2 is to be omitted when $m=0$. Corresponding to the primary wave $e^{i(nt+kx)}$, we find as the expression of the secondary at a great distance from the cylinder,

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[-\frac{k^2c^2}{16}(k^2c^2 - k'^2c^2) \right.$$
$$\left. - k^2c^2 \frac{k'^2 - k^2}{k'^2 + k^2} \cos\theta - \frac{1}{8} k^4c^4 \frac{k^2 - k'^2}{k^2 + k'^2} \cos 2\theta \right]. \quad . \quad (67)$$

The term in $\cos\theta$ is now the leading term; so that the second-

* In (66) c is the magnetic component, and not the radius of the cylinder. So many letters are employed in the electromagnetic theory, that it is difficult to hit upon a satisfactory notation.

ary disturbance approximately vanishes in the direction of the primary electrical displacements, agreeably with what has been proved before. It should be stated here that (67) is not complete to the order k^4c^4 in the term containing $\cos\theta$. The calculation of the part omitted is somewhat tedious in general; but if we introduce the supposition that the difference between k'^2 and k^2 is small, its effect is to bring in the factor $(1-\frac{1}{4}k^2c^2)$.

Extracting the factor (k'^2-k^2), we may conveniently write (67)

$$\psi = -k^2c\,\frac{k'^2-k^2}{k'^2+k^2}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)}\left[\cos\theta - \frac{k'^2c^2+k^2c^2}{16} - \frac{k^2c^2}{8}\cos 2\theta\right], \quad . \quad (68)$$

in which

$$\cos\theta - \frac{k'^2c^2+k^2c^2}{16} - \frac{k^2c^2}{8}\cos 2\theta = \cos\theta - \frac{k'^2c^2-k^2c^2}{16} - \frac{k^2c^2}{4}\cos^2\theta. \quad . \quad . \quad (69)$$

In the directions $\cos\theta=0$, the secondary light is thus not only of high order in kc, but is also of the second order in $(k'-k)$. For the direction in which the secondary light vanishes to the next approximation, we have

$$\tfrac{1}{2}\pi-\theta = \tfrac{1}{16}(k'^2c^2-k^2c^2) = \frac{k^2c^2}{16}\,\frac{K'-K}{K}. \quad . \quad . \quad (70)$$

This corresponds to (61) for the sphere; and is true if kc, $k'c$ be small enough, whatever may be the relation of k' and k. For the cylinder, as for the sphere, the direction is such that the primary light would be bent through an angle *greater* than a right angle.

If we neglect the square of (k'^2-k^2), the complete expression corresponding to (69) is

$$\cos\theta(1-\tfrac{1}{4}k^2c^2)-\tfrac{1}{4}k^2c^2\cos^2\theta = \cos\theta[1-\tfrac{1}{4}k^2c^2-\tfrac{1}{4}k^2c^2\cos\theta].$$

This may be compared with the value obtained by the former method, viz. $\cos\theta\,J_1(2kc\cos\frac{1}{2}\theta)\div kc\cos\frac{1}{2}\theta$, and will be found to agree with it as far as the square of kc.

If we suppose the cylinder to be extremely small, we may confine ourselves to the leading terms in (65) and (67). Let us compare the intensities of the secondary lights emitted in the two cases along $\theta=0$, *i. e.* directly backwards. From (65)

$$\psi \propto \frac{k'^2c^2-k^2c^2}{2},$$

while from (67)

$$\psi \propto -k^2c^2\,\frac{k'^2-k^2}{k'^2+k^2}.$$

The opposition of sign is apparent only, and relates to the different methods of measurement adopted in the two cases. In (65) the primary and secondary disturbances are represented by h/K, but in (67) by the magnetic function c. If we express the solution in the second case in terms of the electric function g, we shall find (see 13) that the ratio of c to g changes sign when we pass from the primary light propagated along $-x$ to the secondary light propagated along $+x$. The actual ratio of amplitudes in the two cases is thus $(k'^2+k^2)/2k^2$, or $(\mathrm{K}'+\mathrm{K})/2\mathrm{K}$. Unless the difference between K′ and K be neglected, the two components of unpolarized light are scattered along this direction in different proportions, that component preponderating in which the electric displacement is parallel to the axis of the cylinder. The secondary light is therefore partially polarized in the plane perpendicular to the axis.

June 1881.

XI. *An Abstract of the Results obtained in a Recalculation of the Atomic Weights. By* FRANK WIGGLESWORTH CLARKE, *S.B., Professor of Chemistry in the University of Cincinnati**.

DURING the past three years I have been engaged upon a recalculation of all the atomic-weight determinations which have been published from the time of Berzelius's earlier investigations down to the present date. My purpose has been to reduce all similar series of experiments to common standards, to calculate the probable error of each series, to combine the results into general means, and then to deduce the atomic weights in such a way that each value should represent a fair average of all the trustworthy estimations. In other words, I have sought to bring together all the vast number of scattered details, and to derive from them a more consistent table of atomic weights than has hitherto been found in chemical literature. My complete work will appear in due time as a separate volume; my present intention is to give merely a summary of my methods, and my conclusions.

Taking hydrogen as unity, I necessarily began with the ratio between it and oxygen. This ratio has been determined accurately in only two ways:—first, by the synthesis of water over copper oxide; and secondly, from the relative density of the two gases. Ignoring earlier inexact experiments, we may consider only the data furnished by Dumas, by Erdmann and Marchand, and by Regnault. From Dumas's nineteen syntheses of water we get for oxygen values ranging

* Communicated by the Author.

where the deposits in the Bay of Bengal are probably the cause of its great depth; and where earthquakes in the intervening regions betray when the faults are establishing themselves which render the rising and the descending areas independent of one another, and allow the denudation on the one side and the deposition on the other to produce each its full effect, without mutual interference.

Of course all compressions and dilatations must be accompanied by other movements within the earth, and at all depths; which may be slow but are no less sure. In fact, there is no material which can resist yielding to differences of pressure, however feeble, if they act for a long time and over a large surface; and such pressures, urging in various directions, must arise both from the compressions and dilatations spoken of above, and from other causes, among which movements of heat and the heterogeneous character of the materials of which the earth consists are prominent. The earth, therefore, is in a state of never-ending change, which to become conspicuous to man would only need to be placed in some kind of kinematograph arrangement which would hurry over millions of years in fractions of a second. These effects mix with and complicate those which have been taken account of in the present paper.

It is interesting to note how the agencies we have been considering would operate upon other bodies of the universe. Events equivalent to denudation and deposition which cause excessively slow movements in our small earth, would act with increased promptness upon such great planets as Jupiter, Saturn, Uranus, and Neptune, and with violence upon bodies that attain the size of the sun and stars. On the other hand, on bodies with the dimensions of the moon they are relatively feeble, and must be very slow in producing any appreciable effect.

XXXIV. *On the Transmission of Light through an Atmosphere containing Small Particles in Suspension, and on the Origin of the Blue of the Sky.* *By* Lord Rayleigh, *F.R.S.**

THIS subject has been treated in papers published many years ago †. I resume it in order to examine more closely than hitherto the attenuation undergone by the primary light on its passage through a medium containing small particles, as dependent upon the number and size of the particles. Closely connected with this is the interesting

* Communicated by the Author.

† Phil. Mag. xli. pp. 107, 274, 447 (1871); xii. p. 81 (1881).

question whether the light from the sky can be explained by diffraction from the molecules of air themselves, or whether it is necessary to appeal to suspended particles composed of foreign matter, solid or liquid. It will appear, I think, that even in the absence of foreign particles we should still have a blue sky *.

The calculations of the present paper are not needed in order to explain the general character of the effects produced. In the earliest of those above referred to I illustrated by curves the gradual reddening of the transmitted light by

* My attention was specially directed to this question a long while ago by Maxwell in a letter which I may be pardoned for reproducing here. Under date Aug. 28, 1873, he wrote:—

"I have left your papers on the light of the sky, &c. at Cambridge, and it would take me, even if I had them, some time to get them assimilated sufficiently to answer the following question, which I think will involve less expense to the energy of the race if you stick the data into your formula and send me the result. . . . .

"Suppose that there are N spheres of density ρ and diameter s in unit of volume of the medium. Find the index of refraction of the compound medium and the coefficient of extinction of light passing through it.

"The object of the enquiry is, of course, to obtain data about the size of the molecules of air. Perhaps it may lead also to data involving the density of the æther. The following quantities are known, being combinations of the three unknowns,

M = mass of molecule of hydrogen;
N = number of molecules of any gas in a cubic centimetre at 0° C. and 760 B.
s = diameter of molecule in any gas:—

Known Combinations.

MN = density.
Ms^2 from diffusion or viscosity.

Conjectural Combination.

$\frac{6M}{\pi s^3}$ = density of molecule.

"If you can give us (i.) the quantity of light scattered in a given direction by a stratum of a certain density and thickness; (ii.) the quantity cut out of the direct ray; and (iii.) the effect of the molecules on the index of refraction, which I think ought to come out easily, we might get a little more information about these little bodies.

"You will see by 'Nature,' Aug. 14, 1873, that I make the diameter of molecules about $\frac{1}{1000}$ of a wave-length.

"The enquiry into scattering must begin by accounting for the great observed transparency of air. I suppose we have no numerical data about its absorption.

"But the index of refraction can be numerically determined, though the observation is of a delicate kind, and a comparison of the result with the dynamical theory may lead to some new information."

Subsequently he wrote, "Your letter of Nov. 17 quite accounts for the observed transparency of any gas." So far as I remember, my argument was of a general character only.

which we see the sun a little before sunset. The same reasoning proved, of course, that the spectrum of even a vertical sun is modified by the atmosphere in the direction of favouring the waves of greater length.

For such a purpose as the present it makes little difference whether we speak in terms of the electromagnetic theory or of the elastic solid theory of light; but to facilitate comparison with former papers on the light from the sky, it will be convenient to follow the latter course. The small particle of volume T is supposed to be small in all its dimensions in comparison with the wave-length (λ), and to be of optical density D′ differing from that (D) of the surrounding medium. Then, if the incident vibration be taken as unity, the expression for the vibration scattered from the particle in a direction making an angle θ with that of *primary vibration* is

$$\frac{D'-D}{D}\frac{\pi T}{r\lambda^2}\sin\theta\cos\frac{2\pi}{\lambda}(bt-r)^*, \quad . \quad . \quad . \quad (1)$$

r being the distance from T of any point along the secondary ray.

In order to find the whole emission of energy from T we have to integrate the square of (1) over the surface of a sphere of radius r. The element of area being $2\pi r^2\sin\theta\,d\theta$, we have

$$\int_0^\pi \frac{\sin^2\theta}{r^2}2\pi r^2\sin\theta\,d\theta = 4\pi\int_0^{\frac{1}{2}\pi}\sin^3\theta\,d\theta = \frac{8\pi}{3};$$

so that the energy emitted from T is represented by

$$\frac{8\pi^3}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

on such a scale that the energy of the primary wave is unity per unit of wave-front area.

The above relates to a single particle. If there be n similar particles per unit volume, the energy emitted from a stratum of thickness dx and of unit area is found from (2) by introduction of the factor $n\,dx$. Since there is no waste of energy on the whole, this represents the loss of energy in the primary wave. Accordingly, if E be the energy of the primary wave,

$$\frac{1}{E}\frac{dE}{dx} = -\frac{8\pi^3 n}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}; \quad . \quad . \quad . \quad (3)$$

whence

$$E = E_0 e^{-hx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

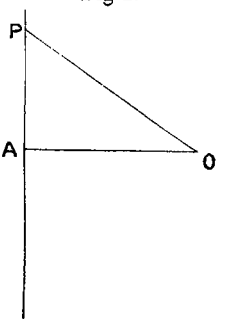
$$h = \frac{8\pi^3 n}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}. \quad . \quad . \quad . \quad (5)$$

* The factor π was inadvertently omitted in the original memoir.

If we had a sufficiently complete expression for the scattered light, we might investigate (5) somewhat more directly by considering the resultant of the primary vibration and of the secondary vibrations which travel in the same direction. If, however, we apply this process to (1), we find that it fails to lead us to (5), though it furnishes another result of interest. The combination of the secondary waves which travel in the direction in question have this peculiarity, that the phases are no more distributed at random. The intensity of the secondary light is no longer to be arrived at by addition of individual intensities, but must be calculated with consideration of the particular phases involved. If we consider a number of particles which all lie upon a primary ray, we see that the phases of the secondary vibrations which issue along this line are all the same.

The actual calculation follows a similar course to that by which Huygens' conception of the resolution of a wave into components corresponding to the various parts of the wave-front is usually verified. Consider the particles which occupy a thin stratum dx perpendicular to the primary ray x. Let AP (fig. 1) be this stratum and O the point where the vibration is to be estimated. If $AP=\rho$, the element of volume is $dx \,.\, 2\pi\rho d\rho$, and the number of particles to be found in it is deduced by introduction of the factor n. Moreover, if $OP=r$, $AO=x$, $r^2=x^2+\rho^2$, and $\rho\, d\rho = r\, dr$. The resultant at O of all the secondary vibrations which issue from the stratum dx is by (1), with $\sin\theta$ equal to unity,

Fig 1.



$$n dx \,.\, \int_x^\infty \frac{D'-D}{D} \frac{\pi T}{r\lambda^2} \cos\frac{2\pi}{\lambda}(bt-r)\, 2\pi r\, dr,$$

or

$$n dx \,.\, \frac{D'-D}{D} \frac{\pi' T}{\lambda} \sin\frac{2\pi}{\lambda}(bt-x). \quad . \quad . \quad . \quad (6)$$

To this is to be added the expression for the primary wave itself, supposed to advance undisturbed, viz., $\cos\frac{2\pi}{\lambda}(bt-x)$, and the resultant will then represent the whole actual disturbance at O as modified by the particles in the stratum dx.

It appears, therefore, that to the order of approximation afforded by (1) the effect of the particles in dx is to modify the phase, *but not the intensity*, of the light which passes

them. If this be represented by

$$\cos\frac{2\pi}{\lambda}(bt-x-\delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

δ is the *retardation* due to the particles, and we have

$$\delta = n\mathrm{T}\,dx\,(\mathrm{D}'-\mathrm{D})/2\mathrm{D}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If μ be the refractive index of the medium as modified by the particles, that of the original medium being taken as unity, $\delta=(\mu-1)dx$, and

$$\mu-1=n\mathrm{T}\,(\mathrm{D}'-\mathrm{D})/2\mathrm{D}. \quad . \quad . \quad . \quad . \quad (9)$$

If μ' denote the refractive index of the material composing the particles regarded as continuous, $\mathrm{D}'/\mathrm{D}=\mu'^2$, and

$$\mu-1=\tfrac{1}{2}n\mathrm{T}(\mu'^2-1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

reducing to

$$\mu-1=n\mathrm{T}(\mu'-1) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in the case where $\mu'-1$ can be regarded as small.

It is only in the latter case that the formulæ of the elastic-solid theory are applicable to light. In the electric theory, to be preferred on every ground except that of easy intelligibility, the results are more complicated in that when $(\mu'-1)$ is not small, the scattered ray depends upon the shape and not merely upon the volume of the small obstacle. In the case of *spheres* we are to replace $(\mathrm{D}'-\mathrm{D})/\mathrm{D}$ by $3(\mathrm{K}'-\mathrm{K})/(\mathrm{K}'+2\mathrm{K})$, where K, K′ are the dielectric constants proper to the medium and to the obstacle respectively*; so that instead of (10)

$$\mu-1=\frac{3n\mathrm{T}}{2}\frac{\mu'^2-1}{\mu'^2+2}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

On the same suppositions (5) is replaced by

$$h=24\pi^3 n\,\frac{(\mu'^2-1)^2}{(\mu'^2+2)^2}\frac{\mathrm{T}^2}{\lambda^4}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

On either theory

$$h=\frac{32\pi^3(\mu-1)^2}{3n\lambda^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

a formula giving the coefficient of transmission in terms of the refraction, and of the *number of particles per unit volume.*

We have seen that when we attempt to find directly from (1) the effect of the particles upon the transmitted primary wave, we succeed only so far as regards the retardation. In

* Phil. Mag. xii. p. 98 (1881). For the corresponding theory in the case of an *ellipsoidal* obstacle, see Phil. Mag. vol. xliv. p. 48 (1897).

order to determine the attenuation by this process it would be necessary to supplement (1) by a term involving

$$\sin 2\pi(bt-r)/\lambda;$$

but this is of higher order of smallness. We could, however, reverse the process and determine the small term in question *à posteriori* by means of the value of the attenuation obtained indirectly from (1), at least as far as concerns the secondary light emitted in the direction of the primary ray.

The theory of these effects may be illustrated by a completely worked out case, such as that of a small rigid and fixed spherical obstacle (radius c) upon which plane waves of sound impinge*. It would take too much space to give full details here, but a few indications may be of use to a reader desirous of pursuing the matter further.

The expressions for the terms of orders 0 and 1 in spherical harmonics of the velocity-potential of the secondary disturbance are given in equations (16), (17), § 334. With introduction of approximate values of γ_0 and γ_1, viz.

$$\gamma_0 + kc = \tfrac{1}{3}k^3c^3, \quad \gamma_1 + kc = \tfrac{1}{2}\pi + \tfrac{1}{6}k^3c^3,$$

we get

$$[\psi_0] + [\psi_1] = -\frac{k^2c^3}{3r}\left(1+\frac{3\mu}{2}\right)\cos k(at-r) + \frac{k^5c^6}{9r}\left(1-\frac{3\mu}{4}\right)\sin k(at-r), \quad . \quad . \quad (15)$$

in which c is the radius of the sphere, and $k=2\pi/\lambda$. This corresponds to the primary wave

$$[\phi] = \cos k(at+x), \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and includes the most important terms from all sources in the multipliers of $\cos k(at-r)$, $\sin k(at-r)$. Along the course of the primary ray ($\mu=-1$) it reduces to

$$[\psi_0] + [\psi_1] = \frac{k^2c^3}{6r}\cos k(at-r) + \frac{7k^5c^6}{36r}\sin k(at-r). \quad . \quad (17)$$

We have now to calculate by the method of Fresnel's zones the effect of a distribution of n spheres per unit volume. We find, corresponding to (6), for the effect of a layer of thickness dx,

$$2\pi n\,dx\{\tfrac{1}{6}kc^3\sin k(at+x) - \tfrac{7}{36}k^4c^6\cos k(at+x)\}. \quad . \quad (18)$$

To this is to be added the expression (16) for the primary wave. The coefficient of $\cos k(at+x)$ is thus altered by the

* 'Theory of Sound,' 2nd ed. § 334.

particles in the layer dx from unity to $(1-\frac{7}{18}k^4c^6\pi n\,dx)$, and the coefficient of $\sin k(at+x)$ from 0 to $\frac{1}{3}kc^3\pi n\,dx$. Thus, if E be the energy of the primary wave,

$$d\mathrm{E}/\mathrm{E} = -\tfrac{7}{9}k^4c^6\pi n\,dx\,;$$

so that if, as in (4), $\mathrm{E}=\mathrm{E}_0e^{-hx}$,

$$h=\tfrac{7}{9}\pi nk^4c^6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The same result may be obtained indirectly from the *first* term of (15). For the whole energy emitted from one sphere may be reckoned as

$$\frac{k^4c^6}{9r^2}\int_{-1}^{+1}2\pi r^2(1+\tfrac{3}{2}\mu)^2d\mu=\frac{7\pi k^4c^6}{9}, \quad . \quad . \quad . \quad (20)$$

unity representing the energy of the primary wave per unit area of wave-front. From (20) we deduce the same value of h as in (19).

The first term of (18) gives the refractivity of the medium. If δ be the retardation due to the spheres of the stratum dx,

$$\sin k\delta=\tfrac{1}{3}kc^3\pi n\,dx,$$

or

$$\delta=\tfrac{1}{3}\pi nc^3dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Thus, if μ be the refractive index as modified by the spheres, that of the original medium being unity,

$$\mu-1=\tfrac{1}{3}\pi nc^3=\tfrac{1}{4}p, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where p denotes the (small) ratio of the volume occupied by the spheres to the whole volume. This result agrees with equations formerly obtained for the refractivity of a medium containing spherical obstacles disposed in cubic order*.

Let us now inquire what degree of transparency of air is admitted by its molecular constitution, *i. e.*, in the absence of all foreign matter. We may take $\lambda=6\times10^{-5}$ centim., $\mu-1=\cdot0003$; whence from (14) we obtain as the distance x, equal to $1/h$, which light must travel in order to undergo attenuation in the ratio $e:1$,

$$x=4{\cdot}4\times10^{-13}\times n. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The completion of the calculation requires the value of n. Unfortunately this number—according to Avogadro's law the same for all gases—can hardly be regarded as known. Maxwell † estimates the number of molecules under standard

* Phil. Mag. vol. xxxiv. p. 499 (1892). Suppose $m=\infty$, $\sigma=\infty$.

† "Molecules," Nature, viii. p. 440 (1873).

conditions as 19×10^{18} per cub. centim. If we use this value of n, we find

$$x = 8{\cdot}3 \times 10^6 \text{ cm.} = 83 \text{ kilometres},$$

as the distance through which light must pass through air at atmospheric pressure before its intensity is reduced in the ratio of 2·7 : 1.

Although Mount Everest appears fairly bright at 100 miles distance as seen from the neighbourhood of Darjeeling, we cannot suppose that the atmosphere is as transparent as is implied in the above numbers; and of course this is not to be expected, since there is certainly suspended matter to be reckoned with. Perhaps the best data for a comparison are those afforded by the varying brightness of stars at various altitudes. Bouguer and others estimate about ·8 for the transmission of light through the entire atmosphere from a star in the zenith. This corresponds to 8·3 kilometres of air at standard pressure. At this rate the transmission through 83 kilometres would be $(\cdot 8)^{10}$, or ·11, instead of $1/e$ or ·37. It appears then that the actual transmission through 83 kilometres is only about 3 times less than that calculated (with the above value of n) from molecular diffraction without any allowance for foreign matter at all. And we may conclude that the light scattered from the molecules would suffice to give us a blue sky, not so very greatly darker than that actually enjoyed.

If n be regarded as altogether unknown, we may reverse our argument, and we then arrive at the conclusion that n cannot be greatly less than was estimated by Maxwell. A lower limit for n, say 7×10^{18} per cubic centimetre, is somewhat sharply indicated. For a still smaller value, or rather the increased individual efficacy which according to the observed refraction would be its accompaniment, must lead to a less degree of transparency than is actually found. When we take into account the known presence of foreign matter, we shall probably see no ground for any reduction of Maxwell's number.

The results which we have obtained are based upon (14), and are as true as the theories from which that equation was derived. In the electromagnetic theory we have treated the molecules as spherical continuous bodies differing from the rest of the medium merely in the value of their dielectric constant. If we abandon the restriction as to sphericity, the results will be modified in a manner that cannot be precisely defined until the shape is specified. On the whole, however, it does not appear probable that this consideration would greatly affect the calculation as to transparency, since the particles

must be supposed to be oriented in all directions indifferently. But the theoretical conclusion that the light diffracted in a direction perpendicular to the primary rays should be *completely* polarized may well be seriously disturbed. If the view, suggested in the present paper, that a large part of the light from the sky is diffracted from the molecules themselves, be correct, the observed incomplete polarization at 90° from the Sun may be partly due to the molecules behaving rather as elongated bodies with indifferent orientation than as spheres of homogeneous material.

Again, the suppositions upon which we have proceeded give no account of *dispersion*. That the refraction of gases increases as the wave-length diminishes is an observed fact; and it is probable that the relation between refraction and transparency expressed in (14) holds good for each wave-length. If so, the falling off of transparency at the blue end of the spectrum will be even more marked than according to the inverse fourth power of the wave-length.

An interesting question arises as to whether (14) can be applied to highly compressed gases and to liquids or solids. Since approximately $(\mu-1)$ is proportional to n, so also is h according to (14). We have no reason to suppose that the purest water is any more transparent than (14) would indicate; but it is more than doubtful whether the calculations are applicable to such a case, where the fundamental supposition, that the phases are entirely at random, is violated. When the volume occupied by the molecules is no longer very small compared with the whole volume, the fact that two molecules cannot occupy the same space detracts from the random character of the distribution. And when, as in liquids and solids, there is some approach to a regular spacing, the scattered light must be much less than upon a theory of random distribution.

Hitherto we have considered the case of obstacles small compared to the wave-length. In conclusion it may not be inappropriate to make a few remarks upon the opposite extreme case and to consider briefly the obstruction presented, for example, by a shower of rain, where the diameters of the drops are large multiples of the wave-length of light.

The full solution of the problem presented by spherical drops of water would include the theory of the rainbow, and if practicable at all would be a very complicated matter. But so far as the direct light is concerned, it would seem to make little difference whether we have to do with a spherical refracting drop, or with an opaque disk of the same diameter.

Let us suppose then that a large number of small disks are distributed at random over a plane parallel to a wave-front, and let us consider their effect upon the direct light at a great distance behind. The plane of the disks may be divided into a system of Fresnel's zones, each of which will by hypothesis include a large number of disks. If α be the area of each disk, and ν the number distributed per unit of area of the plane, the efficiency of each zone is diminished in the ratio $1 : 1 - \nu\alpha$, and, so far as the direct wave is concerned, this is the only effect. The *amplitude* of the direct wave is accordingly reduced in the ratio $1 : 1 - \nu\alpha$, or, if we denote the relative opaque area by m, in the ratio $1 : 1 - m$*. A second operation of the same kind will reduce the amplitude to $(1-m)^2$, and so on. After x passages the amplitude is $(1-m)^x$, which if m be very small may be equated to e^{-mx}. Here mx denotes the whole opaque area passed, reckoned per unit area of wave-front; and it would seem that the result is applicable to any sufficiently sparse random distribution of obstacles.

It may be of interest to give a numerical example. If the unit of length be the centimetre and x the distance travelled, m will denote the projected area of the drops situated in one cubic centimetre. Suppose now that a is the radius of a drop, and n the number of drops per cubic centimetre, then $m = n\pi a^2$. The distance required to reduce the *amplitude* in the ratio $e : 1$ is given by

$$x = 1/n\pi a^2.$$

Suppose that $a = \frac{1}{20}$ centim., then the above-named reduction will occur in a distance of one kilometre ($x = 10^5$) when n is about 10^{-3}, *i.e.* when there is about one drop of one millimetre diameter per litre.

It should be noticed that according to this theory a distant point of light seen through a shower of rain ultimately becomes invisible, not by failure of definition, but by loss of intensity either absolutely or relatively to the scattered light.

* The *intensity* of the direct wave is $1-2m$, and that of the scattered light m, making altogether $1-m$.

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XLII. *On the Scattering of Light by a Cloud of similar small Particles of any Shape and oriented at random.* *By* Lord RAYLEIGH, *O.M., F.R.S.**

FOR distinctness of conception the material of the particles may be supposed to be uniform and non-magnetic, but of dielectric capacity different from that of the surrounding medium; at the same time the results at which we shall arrive are doubtless more general. The smallness is, of course, to be understood as relative to the wave-length of the vibrations.

When the particles are spherical, the problem is simple, as their orientation does not then enter †. If the incident light be polarized, there is no scattered ray in the direction of primary electric vibration, or if the incident light be unpolarized there is complete polarization of the light scattered at right angles to the direction of primary propagation. The consideration of elongated particles shows at once that a want of symmetry must usually entail a departure from the above law of polarization and may be one of the causes, though probably not the most important, of the incomplete polarization of sky-light at 90° from the sun. My son's recent experiments upon light scattered by carefully filtered gases ‡ reveal a decided deficiency of polarization in the light emitted

* Communicated by the Author.

† Phil. Mag. vol. xli. pp. 107, 274, 447 (1871), vol. xii. p. 81 (1881), vol. xlvii. p. 375 (1899); Scientific Papers, vol. i. pp. 87, 104, 518 vol. iv. p. 397.

‡ Roy. Soc. Proc. Feb. 28, 1918.

perpendicularly, and seem to call for a calculation of what is to be expected from particles of arbitrary shape.

As a preliminary to a more complete treatment, it may be well to take first the case of particles symmetrical about an axis, or at any rate behaving as if they were such, for the calculation is then a good deal simpler. We may also limit ourselves to finding the *ratio* of intensities of the two polarized components in the light scattered at right angles, the principal component being that which vibrates parallel to the primary vibrations, and the subordinate component (vanishing for spherical particles) being that in which the vibrations are perpendicular to the primary vibrations. All that we are then concerned with are certain resolving factors, and the integration over angular space required to take account of the random orientations. In virtue of the postulated symmetry, a revolution of a particle about its own axis has no effect, so that in the integration we have to deal only with the direction of this axis. It is to be observed that the system of vibrations scattered by a particle depends upon the direction of primary *vibration* without regard to that of primary *propagation*. In the case of a spherical particle the system of scattered vibrations is symmetrical with respect to this direction and the amplitude of the scattered vibration is proportional to the cosine of the angle between the primary and secondary vibrations. When we pass to unsymmetrical particles, we have first to resolve the primary vibrations in directions corresponding to certain principal axes of the disturbing particle and to introduce separate coefficients of radiation for the different axes. Each of the three component radiations is symmetrical with respect to its own axis, and follows the same law as obtains for the sphere *.

In fig. 1 the various directions are represented by points on a spherical surface with centre O. Thus in the rectangular system XYZ, OZ is the direction of primary vibration, corresponding (we may suppose) to primary propagation parallel to OX. The rectangular system UVW represents in like manner the principal axes of a particle, so that UV, VW, WU are quadrants. Since symmetry of the particle round W has been postulated, there is no loss of generality in taking U upon the prolongation of ZW. As usual, we denote ZW by θ, and XZW by ϕ.

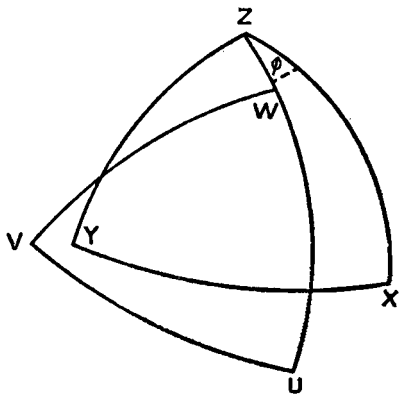
The first step is the resolving of the primary vibration Z in the directions U, V, W. We have

$$\cos \mathrm{ZU} = -\sin\theta, \quad \cos \mathrm{ZV} = 0, \quad \cos \mathrm{ZW} = \cos\theta. \quad . \quad (1)$$

* Phil. Mag. vol. xliv. p. 28 (1897); Sci. Papers, vol. iv. p. 305.

The coefficients, dependent upon the character of the particle, corresponding to U, V, W may be denoted by A, A, C; and we seek the effect along the scattered ray OY,

Fig. 1.



perpendicular to both primary vibrations and primary propagation. The ray scattered in this direction will not be completely polarized, and we consider separately vibrations parallel to Z and to X. As regards the former, we have the same set of factors over again, as in (1), so that the vibration is $A\sin^2\theta + C\cos^2\theta$, reducing to C simply, if A = C. This is the result for a single particle whose axis is at W. What we are aiming at is the aggregate intensity due to a large number of particles with their positions and their axes distributed at random. The *mean* intensity is

$$\int_0^{\frac{1}{2}\pi} \{A + (C-A)\cos^2\theta\}^2 \sin\theta\, d\theta \div \int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta$$

$$= A^2 + \frac{2(C-A)A}{3} + \frac{(C-A)^2}{5} = \tfrac{1}{15}(8A^2 + 3C^2 + 4AC). \quad (2)$$

This represents the intensity of that polarized component of the scattered light along OY whose vibrations are parallel to OZ.

For the vibrations parallel to OX the second set of resolving factors is cos UX, cos VX, cos WX. Now from the spherical triangle UZX,

$$\cos UX = \sin(90° + \theta)\cos\phi = \cos\theta\cos\phi.$$

Also from the triangles VZX, WZX,

$$\cos VX = \cos VZW = \cos(90° + \phi) = -\sin\phi,$$

$$\cos WX = \sin\theta\cos\phi.$$

The first set of factors remains as before. Taking both sets into account, we get for the vibration parallel to X

$$-A \sin\theta \cos\theta \cos\phi + C \cos\theta \sin\theta \cos\phi,$$

the square of which is

$$(C-A)^2 \sin^2\theta \cos^2\theta \cos^2\phi. \quad . \quad . \quad . \quad . \quad (3)$$

The mean value of $\cos^2\phi$ is $\frac{1}{2}$. That of $\cos^2\theta$ is $\frac{1}{3}$ and that of $\cos^4\theta$ is $\frac{1}{5}$, as above, so that corresponding to (2) we have for the mean intensity of the vibrations parallel to X

$$\tfrac{1}{2}(C-A)^2(\tfrac{1}{3}-\tfrac{1}{5}) = \tfrac{1}{15}(C-A)^2. \quad . \quad . \quad . \quad (4)$$

The ratio of intensities of the two components is thus

$$\frac{(C-A)^2}{8A^2+3C^2+4AC}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Two particular cases are worthy of notice. If A can be neglected in comparison with C, (5) becomes simply *one-third.* On the other hand, if A is predominant, (5) reduces to *one-eighth.*

The above expressions apply when the primary light, propagated parallel to X, is completely polarized with vibrations parallel to Z, the direction of the secondary ray being along OY. If the primary light be unpolarized, we have further to include the effect of the primary vibrations parallel to Y. The two polarized components scattered along OY, resulting therefrom, both vibrate in directions perpendicular to OY, and accordingly are both represented by (4). In the case of unpolarized primary light we have therefore to double (4) for the secondary vibrations parallel to X, and to add together (2) and (4) for the vibrations parallel to Z. The latter becomes

$$\tfrac{1}{15}(9A^2+4C^2+2AC),$$

and for the ratio of intensities of the two components

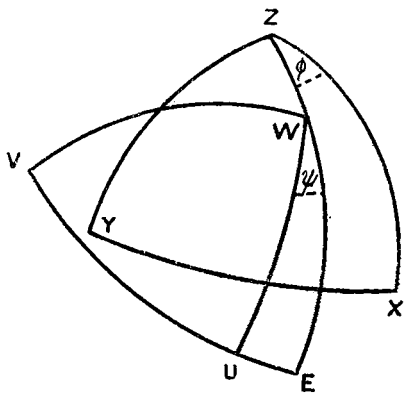
$$\frac{2(C-A)^2}{9A^2+4C^2+2AC}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $A=0$, this ratio is *one-half.*

For a more general treatment, which shall include all forms of particle, we must introduce another angle ψ to

represent the inclination of WU to ZW produced, fig. 2. The direction cosines of either set of axes with respect to

Fig. 2.



the other are given by the formulæ *

$$\left.\begin{aligned} \cos XU &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta \\ \cos YU &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta \\ \cos ZU &= -\sin\theta\cos\psi \end{aligned}\right\}, \quad (7)$$

$$\left.\begin{aligned} \cos XV &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta \\ \cos YV &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta \\ \cos ZV &= \sin\theta\sin\psi \end{aligned}\right\}, \quad (8)$$

$$\left.\begin{aligned} \cos XW &= \sin\theta\cos\phi \\ \cos YW &= \sin\theta\sin\phi \\ \cos ZW &= \cos\theta \end{aligned}\right\}. \quad (9)$$

Supposing, as before, that the primary vibration is parallel to Z, we have as the first set of factors

$$\left.\begin{aligned} \cos ZU &= -\sin\theta\cos\psi, \\ \cos ZV &= \sin\theta\sin\psi, \\ \cos ZW &= \cos\theta \end{aligned}\right\}. \quad (10)$$

For the vibrations propagated along OY which are parallel to Z, we have the same factors over again with coefficients

* See, for example, Routh's 'Rigid Dynamics,' Part I. § 258, 1897. ψ and ϕ are interchanged.

A, B, C as before, and the vibration is expressed by

$$A \sin^2\theta \cos^2\psi + B \sin^2\theta \sin^2\psi + C \cos^2\theta ; \quad . \quad (11)$$

while for the intensity

$$\begin{aligned} I = {} & A^2 \sin^4\theta \cos^4\psi + B^2 \sin^4\theta \sin^4\psi + C^2 \cos^4\theta \\ & + 2AB \sin^4\theta \cos^2\psi \sin^2\psi + 2BC \sin^2\theta \cos^2\theta \sin^2\psi \\ & + 2CA \sin^2\theta \cos^2\theta \cos^2\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

This is for a single particle, and we have now to take the mean for all orientations. The mean value of $\sin^4\psi$, or $\cos^4\psi$, is $\frac{3}{8}$; that of $\sin^2\psi \cos^2\psi$ is $\frac{1}{8}$; and that of $\sin^2\psi$ is $\frac{1}{2}$. The averaging with respect to ψ thus yields

$$\begin{aligned} I = {} & \tfrac{3}{8}(A^2 + B^2) \sin^4\theta + C^2 \cos^4\theta + \tfrac{1}{4} AB \sin^4\theta \\ & + (A + B) C \sin^2\theta \cos^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13) \end{aligned}$$

Again, the mean value of $\sin^4\theta$ is $\frac{8}{15}$, that of $\cos^4\theta$ is $\frac{1}{5}$, and that of $\sin^2\theta \cos^2\theta$ is $\frac{2}{15}$. Thus, finally, the mean value of I over the sphere is given by

$$\text{mean } I = \tfrac{1}{15}\{3(A^2 + B^2 + C^2) + 2(AB + BC + CA)\}. \quad . \quad (14)$$

This refers to the vibrations parallel to Z which are propagated along OY.

For the vibrations parallel to X, the second set of factors is cos XU, cos XV, cos XW, as given above, and the vibration is expressed by

$$\begin{aligned} & -A \sin\theta \cos\psi (-\sin\phi \sin\psi + \cos\phi \cos\psi \cos\theta) \\ & + B \sin\theta \sin\psi (-\sin\phi \cos\psi - \cos\phi \sin\psi \cos\theta) \\ & + C \cos\theta \sin\theta \cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

Accordingly for the intensity

$$\begin{aligned} I = {} & A^2 \sin^2\theta \cos^2\psi (\sin^2\phi \sin^2\psi + \cos^2\phi \cos^2\psi \cos^2\theta \\ & \qquad - 2 \sin\phi \cos\phi \sin\psi \cos\psi \cos\theta) \\ & + B^2 \sin^2\theta \sin^2\psi (\sin^2\phi \cos^2\psi + \cos^2\phi \sin^2\psi \cos^2\theta \\ & \qquad + 2 \sin\phi \cos\phi \sin\psi \cos\psi \cos\theta) \\ & + C^2 \sin^2\theta \cos^2\theta \cos^2\phi \\ & - 2AB \sin^2\theta \sin\psi \cos\psi (\sin^2\phi \sin\psi \cos\psi \\ & \qquad - \cos^2\phi \sin\psi \cos\psi \cos^2\theta + \sin\phi \cos\phi \sin^2\psi \cos\theta \\ & \qquad\qquad - \sin\phi \cos\phi \cos^2\psi \cos\theta) \\ & + 2BC \sin^2\theta \cos\theta \sin\psi \cos\phi (-\sin\phi \cos\psi \\ & \qquad\qquad - \cos\phi \sin\psi \cos\theta) \\ & - 2CA \sin^2\theta \cos\theta \cos\psi \cos\phi (-\sin\phi \sin\psi \\ & \qquad\qquad + \cos\phi \cos\psi \cos\theta). \qquad (16) \end{aligned}$$

In taking the mean with respect to ϕ, the terms which are odd in $\sin\phi$, or $\cos\phi$, disappear, while the mean value of $\sin^2\phi$, or $\cos^2\phi$, is $\frac{1}{2}$. We get for the mean

$$\begin{aligned} I = {} & \tfrac{1}{2}A^2 \sin^2\theta \cos^2\psi (\sin^2\psi + \cos^2\psi \cos^2\theta) \\ & + \tfrac{1}{2}B^2 \sin^2\theta \sin^2\psi (\cos^2\psi + \sin^2\psi \cos^2\theta) \\ & + \tfrac{1}{2}C^2 \sin^2\theta \cos^2\theta \\ & - AB \sin^2\theta \sin\psi \cos\psi \,.\, \sin\psi \cos\psi \sin^2\theta \\ & - BC \sin^2\theta \cos\theta \sin\psi \,.\, \sin\psi \cos\theta \\ & - CA \sin^2\theta \cos\theta \cos\psi \,.\, \cos\psi \cos\theta. \quad . \quad . \quad . \quad (17) \end{aligned}$$

The averaging with respect to ψ now goes as before, and we obtain

$$\begin{aligned} & \tfrac{1}{2}(A^2 + B^2) \sin^2\theta (\tfrac{1}{8} + \tfrac{3}{8}\cos^2\theta) + \tfrac{1}{2}C^2 \sin^2\theta \cos^2\theta \\ & - \tfrac{1}{8}AB \sin^4\theta - \tfrac{1}{2}(A+B)C \sin^2\theta \cos^2\theta; \quad . \quad . \quad . \quad . \quad (18) \end{aligned}$$

and, finally, the averaging with respect to θ gives

$$\begin{aligned} \text{mean } I &= \frac{A^2 + B^2}{16}(1 - \tfrac{1}{3} + \tfrac{6}{15}) + \frac{C^2}{15} - \frac{AB}{15} - \frac{(A+B)C}{15} \\ &= \tfrac{1}{15}\{A^2 + B^2 + C^2 - AB - BC - CA\}. \quad . \quad . \quad . \quad (19) \end{aligned}$$

This represents the intensity of the vibrations parallel to X dispersed along OY, due to primary vibrations parallel to Z. It vanishes, of course, if $A = B = C$; while, if $A = B$ merely, it reduces to (4).

The *ratio* of the two polarized components is

$$\frac{A^2 + B^2 + C^2 - AB - BC - CA}{3(A^2 + B^2 + C^2) + 2(AB + BC + CA)}, \quad . \quad . \quad (20)$$

reducing to (5) when $B = A$.

If the primary light travelling in direction OX is unpolarized, we have also to include primary vibrations parallel to Y. The secondary vibrations scattered along OY are of the same intensity whether they are parallel to Z or to X. They are given by (19), where all that is essential is the perpendicularity of the primary and secondary vibrations. Thus, in order to obtain the effect along OY of unpolarized primary light travelling along OX, we have merely to add (19) to both components. The intensity of

the component vibrating parallel to Z is thus

$$\begin{aligned}&\tfrac{1}{15}\{3(A^2+B^2+C^2)+2(AB+BC+CA)\}\\&+\tfrac{1}{15}\{A^2+B^2+C^2-AB-BC-CA\}\\&=\tfrac{1}{15}\{4(A^2+B^2+C^2)+AB+BC+CA\};\end{aligned} \quad . \quad (21)$$

while that of the component vibrating parallel to X is simply

$$\tfrac{2}{15}\{A^2+B^2+C^2-AB-BC-CA). \quad . \quad . \quad (22)$$

The ratio of the two intensities is

$$\frac{2(A^2+B^2+C^2-AB-BC-CA)}{4(A^2+B^2+C^2)+AB+BC+CA}, \quad . \quad . \quad (23)$$

reducing to (6) when B=A.

It may be observed that, since (21)=(14)+(19), we obtain the same intensity whether we use a polarizer transmitting vibrations parallel to Z and no analyser, or whether we use an analyser transmitting vibrations parallel to Z and no polarizer.

If neither polarizing nor analysing apparatus is employed, we may add (21) and (22), thus obtaining

$$\tfrac{1}{15}[6(A^2+B^2+C^2)-AB-BC-CA]. \quad . \quad . \quad (24)$$

When the particles are supposed to be of uniform quality, with a specific inductive capacity K′ as compared with K for the undisturbed medium, and to be of *ellipsoidal* form with semi-axes a, b, c, we have

$$A^{-1}:B^{-1}:C^{-1}=1+\frac{K'-K}{4\pi K}L:1+\frac{K'-K}{4\pi K}M:1+\frac{K'-K}{4\pi K}N,$$

where . . . (25)

$$L=2\pi abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{1/2}(c^2+\lambda)^{1/2}}, \quad . \quad (26)$$

with similar expressions for M and N.

If the ellipsoid be of revolution the case is simplified *. For example, if it be of the elongated or ovary form with eccentricity e,

$$a=b=c\sqrt{(1-e^2)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$L=M=2\pi\left\{\frac{1}{e^2}-\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad . \quad . \quad (28)$$

$$N=4\pi\left\{\frac{1}{e^2}-1\right\}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\}. \quad . \quad (29)$$

* See the paper of 1897.

For the sphere ($e=0$)

$$L=M=N=\frac{4\pi}{3}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In the case of a very elongated ovoid, L and M approximate to the value 2π, while N approximates to the form

$$N=4\pi\frac{a^2}{c^2}\left(\log\frac{2c}{a}-1\right), \quad . \quad . \quad . \quad . \quad (31)$$

vanishing when $e=1$. It appears that, when K'/K is finite, mere elongation does not suffice to render A and B negligible in comparison with C. The limiting value of C : A is in fact $\frac{1}{2}(1+K'/K)$. If, however, as for a perfectly conducting body, $K'=\infty$, then C becomes paramount, and the simplified values already given for this case acquire validity*.

Another question which naturally presents itself is whether a want of equality among the coefficients A, B, C interferes with the relation between attenuation and refractive index, explained in my paper of 1899 †. The answer appears to be in the affirmative, since the attenuation depends upon $A^2+B^2+C^2$, while the refractive index depends upon $A+B+C$, so that no simple relation obtains in general. But it may well be that in cases of interest the disturbance thus arising is not great.

The problem of an ellipsoidal particle of uniform dielectric quality can be no more than illustrative of what happens in the case of a molecule; but we may anticipate that the general form with suitable values of A, B, C still applies, except it may be under special circumstances where resonance occurs and where the effective values of the coefficients may vary greatly with the wave-length of the light.

* But the particle must still be small relatively to the wave-length *within the medium* of which it is composed.

† An equivalent formula was given by Lorenz in 1890, 'Œuvres Scientifiques,' t. i. p. 496, Copenhagen, 1898. See also Schuster's 'Theory of Optics,' 2nd ed. p. 326 (1909).

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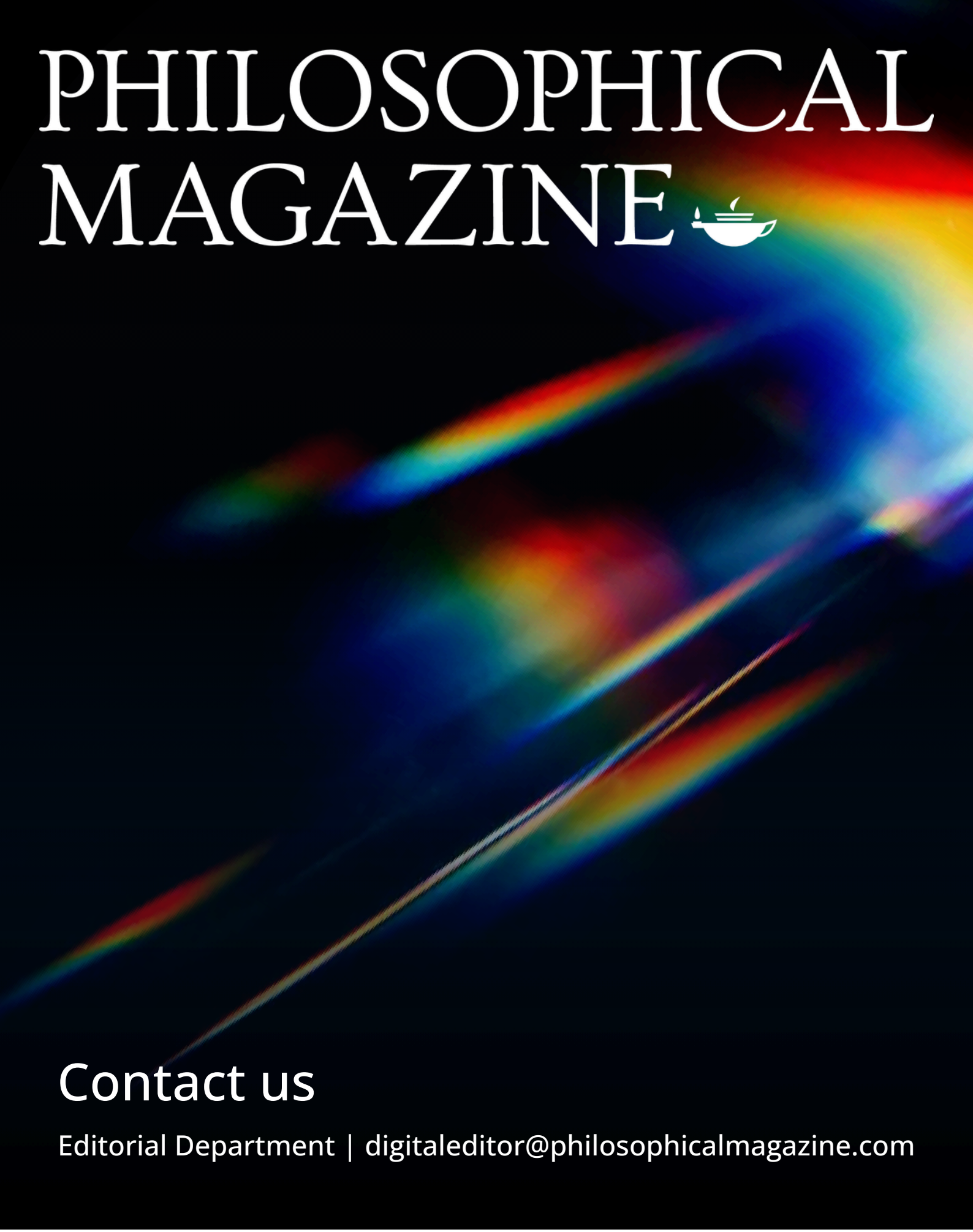

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